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Thermoelastic equilibrium of piecewise homogeneous solids with thin inclusions

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Abstract An effective method of modeling the presence of thin inclusions of arbitrary physical nature in bodies is discussed. Using this method, the plane thermoelastic problem for two bounded dissimilar semi-planes with thin heat-active interface inclusions is reduced to two separate systems of singular integral equations. The concept of generalized stress-intensity factors is introduced and their dependence on the material characteristics and several methods of thermal loading are analyzed.

Keywords Interaction conditions · Stress-intensity factor · Temperature-gradient intensity factor · Thermoelasticity · Thin inclusion

Nomenclature

PCCDD	The principle of conjugation of continua with different dimensions
JF	Jump functions
CPMF	Constituents of physical and mechanical fields
IC	Interaction conditions
SIE	System of integral equations
SSIE	System of singular integral equations
HII	Heat-insulated inclusion
DI	Diathermic inclusion
GTFIF	Gradients of the temperature-field intensity factors
GSIF	Generalized stress-intensity factors
f_r	Jump functions

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T	Temperature
$T_{,n} \equiv \partial T/\partial n$,	
q_x, q_y, q_n	Heat fluxes
$\sigma_{kj}(\xi), u_k(\xi)$	Stresses, displacements
$L'_p = [a_p^-;\ a_p^+]$	Line, modeling the presence of a thin inclusion
$2h(x)$	Inclusion width
δ	Plate thickness
$q_{\mathrm{B}}(x, y, z)$	Specific density of heat sources
E	Young's modulus
ν	Poisson's ratio
α_T	Coefficient of thermal expansion of the material
λ_k, λ_B	Heat-conduction coefficients
α_{yk}	Coefficients of heat emission or values reverse to coefficients of heat resistance
$N_x(w), N_{xy}(w)$,	
$U(w), V(w)$	Stresses and displacements at the inclusion tips
$Q_x(w)$	Heat flux at the inclusion tips
$M(w)$	Moments at the inclusion tips
$\varepsilon_{\mathrm{B}}^w$	Rigid turn of the inclusion
$\Phi_T(z)$	Complex potential of the temperature field
$\Phi_k(z),\ \Psi_k(z)$	Complex Kolosov–Muskhelishvili potentials
$k_1^{\pm}, k_2^{\pm}$	Gradients of the temperature-field intensity factors
$K_{1,m}^{\pm},\ K_{2,m}^{\pm}$	Generalized stress-intensity factors
$\langle\varphi\rangle_h = \varphi(x, -h) + \varphi(x, h)$,	$[\varphi] = \varphi_1^-(x) - \varphi_2^+(x)$,
$[\varphi]_h = \varphi(x, -h) - \varphi(x, h)$	$\langle\varphi\rangle = \varphi_1^-(x) + \varphi_2^+(x)$
superscripts "+", "−"	Denotes the boundary values of the *functions* on the upper and the lower inclusion borders with respect to width
a hat "^"	Marks the disturbed constituents of the fields
superscript "∘"	Marks the CPMF in the corresponding problem without any inclusion
subscript "B"	Denotes the terms CPMF inside the inclusion
superscript "b"	Marks the biharmonic part of CPMF in the case of the generalized plane temperature field.

1 Introduction

Thin inclusions, i.e., inclusions with one dimension smaller than the rest, are widespread as imperfections of material structures and used as stiffening elements in constructions. Similar problems of thin inclusions arise when investigating phase precipitation, when analyzing composite materials with layered and ribbonlike armature which has more advantages in comparison with fibrous composites, when studying split filling with liquid or solid.

It has been noted [1,2] that for solving the problem of heat transfer and thermoelasticity for elastic bodies with thin inclusions, it is possible to select five main approaches of analysis: *general theoretic*—to consider a thin inclusion of arbitrary form, and then to decrease one of its sizes; *numerical*—to apply direct numerical methods; *experimental*—to use experimental methods; *asymptotical*—to consider in detail the stresses and displacements directly near the vicinity of heterogeneities and interface of materials by asymptotic methods; *new theories of imperfect contact*—to develop a specific theory that will enable to solve the proper problems rather than simply taking into account the effect of a defect's small thickness.

The idea of the last of these, which is one of the most productive approaches, is based on the principle of conjugation of continua with different dimensions [3–20]. An object is eliminated from consideration and its influence results in the appearance of jumps in the temperature, heat fluxes, displacement vectors and stresses in the matrix. Then stresses and other characteristics in an arbitrary point of the solid are determined by the problem geometry, material properties, external loading and jump functions. The mathematical model of an inclusion is given as the *interaction conditions* that are equivalent to the conditions of imperfect contact between the matrix surfaces adjacent to the inclusion.

2 Jump-function method

The idea behind the jump-function method is based on the application of two principal propositions: the principle of conjugation of continua with different dimensions (PCCDD) and existing interaction conditions (IC) between thin inclusions and their surroundings. The essence of the PCCDD lies in the substitution of a thin inclusion of volume V (area in the two-dimensional case) by some surface S (curve L in two-dimensional problems) of temperature, stress and displacement discontinuities in the surrounding media. An inclusion as a geometrical object is eliminated from consideration, and its influence results in the appearance in a matrix of jump functions (JF), f_r, of some physical and mechanical fields (temperature T, heat fluxes $T_{,n} \equiv \partial T/\partial n$, components of the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, stress vector $\mathbf{t}_n = (t_{n_1}, t_{n_2}, t_{n_3})$) and the like, while passing across the surface S (curve L) [20]. The JF are functions of the surface S (curve$L$) co-ordinates.

The selection of the amount of physical and mechanical content of the JF taken into consideration must represent those effects which are generated by the presence of the examined type of thin heterogeneities in the considered class of environment and to provide a simple and synonymous determination of all constituents of physical and mechanical fields (CPMF) taken into consideration in an arbitrary point ξ of the medium beyond the region filled by inclusions. Thus, not only the JF but also the external loading, the geometry of problem and the material properties must be taken into account.

If some external loading acts in a matrix there, it is necessary to supply corresponding terms with the homogeneous solution $\sigma^0_{kj}(\xi)$, $u^0_k(\xi)$, $T^0(\xi)$, i.e., the expressions describing given parameters under the given load and without inclusions (jump surfaces), in the equations for stress, strain and displacements. Thus the solution can be given as the superposition

$$\begin{aligned} &\sigma_{kj}(\xi) = \hat{\sigma}_{kj}(\xi,\ f_r) + \sigma^0_{kj}(\xi), \quad u_k(\xi) = \hat{u}_k(\xi,\ f_r) + u^0_k(\xi), \\ &T(\xi) = \hat{T}(\xi,\ f_r)\ + T^0(\xi), \end{aligned} \tag{2.1}$$

where a hat marks the disturbed constituents of the fields, obtained only as a result of JF influence, and the superscript “∘” marks the CPMF in the corresponding problem without any inclusions (homogeneous solution).

In some problems (opening with wedge, distribution of dislocations, forces, heat sources in surfaces etc.) all or part of the JF are known. However, in most cases the JF are unknown. Then, for the determination of originally unknown JF, one uses interaction conditions (IC) which reflect the dependence $F_j\left(T^{\pm}_{\mathrm{B}},\ T^{\pm}_{\mathrm{B},n},\ \mathbf{t}^{\pm}_{n\mathrm{B}},\ \mathbf{u}^{\pm}_{\mathrm{B}}\right) = 0$ between the temperature $T^{\pm}_{\mathrm{B}}$, the heat flux $T^{\pm}_{\mathrm{B},n}$ through the interface, the displacements $\mathbf{u}^{\pm}_{\mathrm{B}}$ and the stress vectors $\mathbf{t}^{\pm}_{n\mathrm{B}}$ inside the thin inclusion, where superscripts “+” and “−” denote boundary values of these functions on the upper and the lower inclusion borders, respectively. These equations constitute the mathematical inclusion model.

In thermoelasticity the number of these equations must correspond to the number of unknown JF (in the most general three-dimensional form there should be eight of these, namely three for jumps in the stresses, three for jumps in the displacements and two for jumps in the temperature and heat flux).

Using boundary conditions on the interface (perfect or imperfect) and transferring the value of CPMF from matrix-inclusion interface to jump surface S (curve$L$), one can define the mathematical inclusion model on the inclusion-matrix interface:

$$\Psi_j\left(T^{\pm},\ T^{\pm}_{,n},\ \mathbf{t}^{\pm}_n,\ \mathbf{u}^{\pm}\right) = 0, \quad t^{\pm}_{nk} = \sigma^{\pm}_{kj} n_j. \tag{2.2}$$

The interaction conditions (2.2) are, per se, conditions of imperfect contact between the opposite matrix surfaces adjacent to the inclusion.

If one defines boundary values of the CPMF on the interface from (2.1) and substitutes them in the IC (2.2), a system of integral equations (SIE) for the determination of the jump functions $\Phi_j(f_r) = 0$ is obtained. On the basis of PCCDD (2.1) the solution of this system determines the CPMF in an arbitrary point of the matrix. The application of the mathematical inclusion model (2.2) allows determining the CPMF inside the inclusion.

It should be noted that PCCDD and IC can be examined independently. PCCDD (as an external problem) can be applied without taking into consideration the mechanical properties of an inclusion and is determined exclusively by matrix properties: if the expressions (2.1) are created by the given type of matrix, such a dependence can be applicable for arbitrary IC as in (2.2), developed for cracks, rigid layers or for various models of elastic, elastic–plastic, thin liquid inclusions.

On the other hand, the matrix type is not important for the creation of IC (internal problem): it is no more than some abstract continuum. Requirements for an IC are: (1) the number of IC must be equal to the number of JF; (2) they must be simple enough to solve the obtained SIE; (3) they must be sufficiently adequate to reflect the essential characteristics of inclusion behavior.

3 Method of thin elastic heat-active inclusion design

The thin-object method, as presented by the authors, is based on the idea of integration with respect to volume (size) for determining expressions that describe the physical and mechanical state of inclusion material, taking account of the smallness of one of the linear sizes of the inclusion.

Let us consider an inclusion of small width $2h(x)$, symmetric with respect to the middle line and situated along $L'_p = [a_p^-;\ a_p^+]$ (Fig. 1) on the interface of two half-planes of thickness 2δ (for plane temperature field and plane deformation $\delta \to \infty$).

The thermoelastic equilibrium of an elementary volume, abstracted from the elastic isotropic inclusion, is described by the equations of heat balance and elastic equilibrium, for the displacements u_x, u_y, stresses σ_{xx}, σ_{yy}, σ_{xy} and temperature T_B:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{q_n}{\delta} + \frac{q_B(x, y, z)}{2\delta} = 0, \tag{3.1}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{3.2}$$

$$\begin{gathered} \varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx} - \nu_* \sigma_{yy}}{E_*} + \alpha_{T*} T_B, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\sigma_{yy} - \nu_* \sigma_{xx}}{E_*} + \alpha_{T*} T_B, \\ \gamma_{xy} \equiv 2\varepsilon_{xy} = \frac{2(1+\nu_*)\sigma_{xy}}{E_*}, \quad \gamma_{xy} \equiv 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \end{gathered} \tag{3.3}$$

where q_x, q_y, q_n are the heat fluxes in the directions x, y, n; $\mathbf{n}$ is the external normal to the lateral surfaces of the inclusion; $q_{\mathrm{B}}(x, y, z)$ denotes the specific density of heat sources; $E_* = E/\left(1-\nu^2\right)$, $\nu_* = \nu/(1-\nu)$, $\alpha_{T*} =$

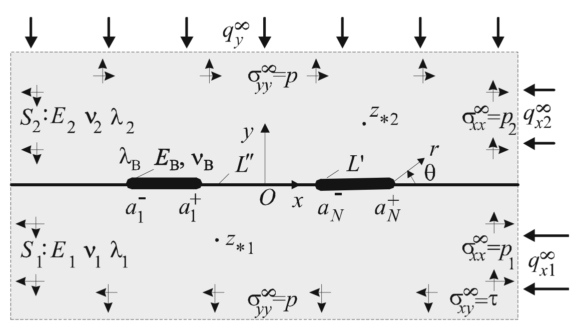

Fig. 1 Bonded half-planes with interface inclusions under different loading

$(1+\nu)\,\alpha_T$ for plane deformation; $E_* = E$, $\nu_* = \nu$, $\alpha_{T*} = \alpha_T$ for generalized plane stress; E, ν, α_T are Young's modulus, the Poisson's ratio and the coefficient of thermal expansion of the material, respectively.

In this case the single equation of displacements compatibility looks like:

$$\Delta\left(\sigma_{xx}+\sigma_{yy}\right)+\alpha_{T*}E_*\Delta T_B=0. \tag{3.4}$$

Let the heat transfer on lateral surfaces ($z=\pm\delta$) satisfy the generalized law $q_n = F\left(T_{\mathrm{B}}, T_C, \alpha, \beta, \ldots\right)$, where $T_{\mathrm{B}}, T_C, \alpha, \beta, \ldots$ denote the temperature of the inclusion, the temperature and the physical parameters of the surroundings, respectively. Then one can speak about a generalized plane temperature field, and taking into account the Fourier law, we have:

$$\begin{aligned}&\lambda_{\mathrm{B}}\Delta T_{\mathrm{B}}(x,y)-F\left(T_{\mathrm{B}},T_C,\alpha,\beta,\ldots\right)/\delta-Q_{\mathrm{B}}(x,y)=0,\\&\{T_{\mathrm{B}}(x,y),\,Q_{\mathrm{B}}(x,y)\}=\frac{1}{2\delta}\int_{-\delta}^{\delta}\{T_{\mathrm{B}}(x,y,z),\ q_{\mathrm{B}}(x,y,z)\}\mathrm{d}z.\end{aligned} \tag{3.5}$$

Here T_{B}, Q_{B} denote the temperature and density of internal heat sources averaged with respect to thickness.

In particular, for heat transfer according to Newton's law $f\left(T_{\mathrm{B}}, T_C, \alpha_{\mathrm{B}}\right)=\alpha_{\mathrm{B}}\left(T_{\mathrm{B}}-T_C\right)$ at $z=\pm\delta$ and Eq. (3.5) becomes [12]

$$\lambda_B\Delta T(x,y)-\frac{\alpha_B}{\delta}\left[T(x,y)-T_C\right]=-\frac{Q_B(x,y)}{2\delta}. \tag{3.6}$$

For a plane temperature field $(\delta\to\infty)\,T_{\mathrm{B}}(x,y,z) = T_{\mathrm{B}}(x,y)$ and the equation of heat conduction (3.6) becomes harmonic.

Integrating (3.1), (3.2) with respect to the width y and to x in the domain $\left[a_p^-;\ x\right]$, we obtain:

$$\lambda_{\mathrm{B}}\int_{-h}^{h}\frac{\partial T_{\mathrm{B}}(x,y)}{\partial x}\mathrm{d}y-Q_x\left(a_p^-\right)-\lambda_{\mathrm{B}}\int_{a_p^-}^{x}\left[\frac{\partial T_{\mathrm{B}}}{\partial y}\right]_h\mathrm{d}\xi-\frac{2h}{\delta}\int_{a_p^-}^{x}F^C(\xi)\mathrm{d}\xi+2h\int_{a_p^-}^{x}Q_{\mathrm{B}}^C(\xi)\mathrm{d}\xi=0, \tag{3.7}$$

$$\int_{-h}^{h}\sigma_{xx\mathrm{B}}(x,y)\,\mathrm{d}y-\int_{-h}^{h}\sigma_{xx\mathrm{B}}\left(a_p^-,y\right)\mathrm{d}y-\int_{a_p^-}^{x}\left[\sigma_{xy\mathrm{B}}\right]_h\mathrm{d}\xi=0, \tag{3.8}$$

$$\int_{-h}^{h}\sigma_{xy\mathrm{B}}(x,y)\,\mathrm{d}y-\int_{-h}^{h}\sigma_{xy\mathrm{B}}\left(a_p^-,y\right)\mathrm{d}y-\int_{a_p^-}^{x}\left[\sigma_{yy\mathrm{B}}\right]_h\mathrm{d}\xi=0, \tag{3.9}$$

where $Q_x(w)=\int_{-h(w)}^{h(w)}\lambda_{\mathrm{B}}\frac{\partial T_B(w,y)}{\partial x}\mathrm{d}y,\ \left(w=\{a_p^-,\ a_p^+\}\right)$, are the heat fluxes at the tips of the inclusion; also

$$\varphi^C(x)=\frac{1}{2h}\int_{-h}^{h}\varphi(x,y)\,\mathrm{d}y,\quad \langle\varphi\rangle_h=\varphi(x,-h)+\varphi(x,h),\quad [\varphi]_h=\varphi(x,-h)-\varphi(x,h).$$

Using the expansion of functions into Taylor series with respect to y near the points $(x,\pm h)$ and keeping terms of order not higher than h^2 in the equations, we get a system of mathematical model equations for a thin elastic heat-conductive inclusion in the case of a plane problem:

$$\begin{cases} -\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\sigma_{xy\mathrm{B}}\right]_h \mathrm{d}\xi+\int_{a_p^-}^{x}\left[\frac{\partial u_{y\mathrm{B}}}{\partial x}\right]_h \mathrm{d}\xi+\frac{h}{E_{\mathrm{B}}}\left\langle\sigma_{yy\mathrm{B}}\right\rangle_h+\alpha_{T\mathrm{B}}h\left\langle T_{\mathrm{B}}\right\rangle_h \\ -\frac{h^2}{E_{\mathrm{B}}}\frac{\partial}{\partial x}\left[\sigma_{xy\mathrm{B}}\right]_h+\frac{\alpha_{T\mathrm{B}}h^2}{\lambda_{\mathrm{B}}}\left[\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial y}\right]_h-\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}N_x\left(a_p^-\right)+V\left(a_p^-\right)=0, \\ -\frac{1}{E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\sigma_{xy\mathrm{B}}\right]_h \mathrm{d}\xi+h\left\langle\frac{\partial u_{x\mathrm{B}}}{\partial x}\right\rangle_h+\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h\left\langle\sigma_{yy\mathrm{B}}\right\rangle_h-\alpha_{T\mathrm{B}}h\left\langle T_{\mathrm{B}}\right\rangle_h \\ -h^2\frac{\partial}{\partial x}\left[\frac{\partial u_{y\mathrm{B}}}{\partial x}\right]_h+\frac{2+\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h^2\frac{\partial}{\partial x}\left[\sigma_{xy\mathrm{B}}\right]_h-\frac{\alpha_{T\mathrm{B}}h^2}{\lambda_{\mathrm{B}}}\left[\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial y}\right]_h-\frac{1}{E_{\mathrm{B}}}N_x\left(a_p^-\right)=0, \\ -\int_{a_p^-}^{x}\left[\frac{\partial u_{x\mathrm{B}}}{\partial x}\right]_h \mathrm{d}\xi+\frac{1}{\nu_{\mathrm{B}}E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\sigma_{yy\mathrm{B}}\right]_h \mathrm{d}\xi-\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h\left\langle\sigma_{xy\mathrm{B}}\right\rangle_h \\ +\frac{1+\nu_{\mathrm{B}}}{\nu_{\mathrm{B}}}\alpha_{T\mathrm{B}}\int_{a_p^-}^{x}\left[T_{\mathrm{B}}\right]_h \mathrm{d}\xi+\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}N_{xy}\left(a_p^-\right)-\frac{1}{\nu_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\frac{\partial u_{y\mathrm{B}}}{\partial y}\right]_h \mathrm{d}\xi \\ +\frac{h^2}{E_{\mathrm{B}}}\frac{\partial}{\partial x}\left[\sigma_{yy\mathrm{B}}\right]_h+\alpha_{T\mathrm{B}}h^2\frac{\partial}{\partial x}\left[T_{\mathrm{B}}\right]_h-h^2\frac{\partial}{\partial x}\left[\frac{\partial u_{y\mathrm{B}}}{\partial y}\right]_h=0, \\ -\int_{a_p^-}^{x}\left[\frac{\partial u_{x\mathrm{B}}}{\partial x}\right]_h \mathrm{d}\xi-\frac{1}{G_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\sigma_{yy\mathrm{B}}\right]_h \mathrm{d}\xi+h\left\langle\frac{\partial u_{y\mathrm{B}}}{\partial x}\right\rangle_h+h^2\frac{\partial}{\partial x}\left[\frac{\partial u_{y\mathrm{B}}}{\partial y}\right]_h=V'\left(a_p^-\right)\quad\left(x\in L_p'\right), \end{cases} \tag{3.10}$$

where $\left\{N_x(w), N_{xy}(w), U'(w), V'(w)\right\}$ are the stresses and displacements at the inclusion tips. Further

$$\begin{cases} \lambda_{\mathrm{B}}\left[T_{\mathrm{B}}\right]_h+h\left\langle\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial y}\right\rangle_h+\frac{h^2}{2}\left\{-\lambda_{\mathrm{B}}\frac{\partial^2}{\partial x^2}\left[T_{\mathrm{B}}\right]_h+[F]_h/\delta+\left[Q_{\mathrm{B}}\right]_h\right\}=0, \\ \lambda_{\mathrm{B}}h\left\langle\frac{\partial T_{\mathrm{B}}}{\partial x}\right\rangle_h+h^2\frac{\partial}{\partial x}\left[\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial y}\right]_h-\int_{a_p^-}^{x}\left[\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial y}\right]_h \mathrm{d}\xi-\frac{2h}{\delta}\int_{a_p^-}^{x}F^C(\xi)\,\mathrm{d}\xi \\ +2h\int_{a_p^-}^{x}Q_{\mathrm{B}}^C(\xi)\,\mathrm{d}\xi-Q_x\left(a_p^-\right)=0. \end{cases} \tag{3.11}$$

If we keep terms of order not higher than h in (3.10)–(3.11), then this version of the equations will differ from the equations of the "intermediate layer" model [9,11] only by coefficients and terms that take into account the heat activity of the inclusion. It is possible to take into account also terms of order h^3 etc and obtain more exact, but substantially more difficult models for the inclusion.

Let us suppose now that the thermal contact between inclusion and matrix along the whole thickness is imperfect ($x \in L_p'$):

$$\lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}(x,y)}{\partial y}\pm\alpha_{yk}\left\{T_{\mathrm{B}}(x,\pm h)-T(x,\pm h)\right\}=0;\quad \lambda_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}(x,y)}{\partial y}=\lambda_k\frac{\partial T(x,\pm h)}{\partial y}, \tag{3.12}$$

where $T(x, y)$ is the temperature of the matrix; λ_k $(k = 1, 2)$ are the coefficients of heat conduction of the matrix materials; α_{yk} are heat-transfer coefficients (for fluid or gas inclusion) or values opposite to coefficients of heat resistance (for inclusion of crystalloid structure). For $\alpha_{yk} = 0$ in (3.12), we obtain the partial case of a heat-insulated inclusion (crack). For $\alpha_{yk} \to \infty$ the conditions (3.12) describe perfect thermal contact.

The mechanical contact between inclusion and matrix will be considered as perfect:

$$\begin{gathered} \sigma_{yy\mathrm{B}}(x,\pm h)=\sigma_{yyk}(x,\pm h),\quad \sigma_{xy\mathrm{B}}(x,\pm h)=\sigma_{xyk}(x,\pm h), \\ u_{x\mathrm{B}}'(x,\pm h)=u_{xk}'(x,\pm h),\quad u_{y\mathrm{B}}'(x,\pm h)=u_{yk}'(x,\pm h)+\varepsilon_{\mathrm{B}},\quad k=\begin{Bmatrix}2\\1\end{Bmatrix}\ \left(x\in L_p'\right), \end{gathered} \tag{3.13}$$

where $\sigma_{yyk}, \sigma_{xyk}, u_{xk}, u_{yk}$ $(k = 1, 2)$ denote the stresses and displacements in the matrix at the upper $(k = 2)$ and lower $(k = 1)$ bounds of contact, respectively; ε_{B} is the fixed rotation of the inclusion.

Taking into account the thinness of the inclusion, we may use:

$$\frac{\partial u_{y\mathrm{B}}(x,h)}{\partial y}=\frac{\partial u_{y\mathrm{B}}(x,-h)}{\partial y}\quad\text{or}\quad\left[\frac{\partial u_{y\mathrm{B}}}{\partial y}\right]_h=0. \tag{3.14}$$

The use of the boundary conditions (3.12)–(3.13), while taking into account (3.14) in (3.10), (3.11) of the mathematical model of the inclusion, generates *the thermomechanical interaction conditions* of a thin elastic heat-active inclusion-layer with matrix:

$$\left\{\begin{array}{l} -\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\hat{\sigma}_{xy}\right]\mathrm{d}\xi+\int_{a_p^-}^{x}\left[u'_y\right]\mathrm{d}\xi+\frac{h}{E_{\mathrm{B}}}\left\langle\hat{\sigma}_{yy}\right\rangle-\frac{h^2}{E_{\mathrm{B}}}\frac{\partial}{\partial x}\left[\hat{\sigma}_{xy}\right]=F_1(x),\\ -\frac{1}{E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\hat{\sigma}_{xy}\right]\mathrm{d}\xi+h\left\langle\hat{u}'_x\right\rangle+\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h\left\langle\hat{\sigma}_{yy}\right\rangle+\frac{2+\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h^2\frac{\partial}{\partial x}\left[\hat{\sigma}_{xy}\right]-h^2\frac{\partial}{\partial x}\left[\hat{u}'_y\right]=F_2(x),\\ -\int_{a_p^-}^{x}\left[\hat{u}'_x\right]\mathrm{d}\xi+\frac{1}{\nu_{\mathrm{B}}E_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\hat{\sigma}_{yy}\right]\mathrm{d}\xi-\frac{\nu_{\mathrm{B}}}{E_{\mathrm{B}}}h\left\langle\hat{\sigma}_{xy}\right\rangle+\frac{h^2}{E_{\mathrm{B}}}\frac{\partial}{\partial x}\left[\hat{\sigma}_{yy}\right]=F_3(x),\\ -\int_{a_p^-}^{x}\left[\hat{u}'_x\right]\mathrm{d}\xi-\frac{1}{G_{\mathrm{B}}}\int_{a_p^-}^{x}\left[\hat{\sigma}_{yy}\right]\mathrm{d}\xi+h\left\langle\hat{u}'_y\right\rangle=F_4(x), \end{array}\right. \tag{3.15}$$

$$\left\{\begin{array}{l} \frac{\lambda_{\mathrm{B}}}{h}\left(1-\frac{h^2}{2}\frac{\partial^2}{\partial x^2}\right)\left\{\left[\hat{t}\right]+\left\langle\frac{\lambda}{\alpha_y}\frac{\partial\hat{t}}{\partial y}\right\rangle\right\}+\left\langle\lambda\frac{\partial\hat{t}}{\partial y}\right\rangle+\frac{h}{2\delta}\left[F\right]=F_5(x),\ (x\in L'_p),\\ \lambda_{\mathrm{B}}h\frac{\partial}{\partial x}\left\{\left\langle\hat{t}\right\rangle+\left[\frac{\lambda}{\alpha_y}\frac{\partial\hat{t}}{\partial y}\right]\right\}+h^2\frac{\partial}{\partial x}\left[\lambda\frac{\partial\hat{t}}{\partial y}\right]-\int_{a}^{x}\left[\lambda\frac{\partial\hat{t}}{\partial y}\right]\mathrm{d}\xi-\frac{2h}{\delta}\int_{a_p^-}^{x}F^C\left(\xi\right)\mathrm{d}\xi=F_6(x), \end{array}\right.$$

where $F_m\left(x\right)$ $(m=1,2,3,4)$ are given in Appendix 1.

Substituting $x=a_p^+$ in (3.15), we obtain the integral conditions describing the heat balance and elastic equilibrium in the inclusion region:

$$\int_{L'_p}\left[\lambda\frac{\partial T}{\partial y}\right]_h\mathrm{d}\xi=Q_x\left(a_p^+\right)-Q_x\left(a_p^-\right)-\frac{2h}{\delta}\int_{L'_p}f^C\left(\xi\right)\mathrm{d}\xi+2h\int_{L'_p}Q_B^C\left(\xi\right)\mathrm{d}\xi, \tag{3.16}$$

$$\int_{a_p^-}^{a_p^+}\left[\sigma_{yy}\right]_h\mathrm{d}\xi=N_{xy}\left(a_p^+\right)-N_{xy}\left(a_p^-\right),\quad\int_{a_p^-}^{a_p^+}\left[\sigma_{xy}\right]_h\mathrm{d}\xi=N_x\left(a_p^+\right)-N_x\left(a_p^-\right). \tag{3.17}$$

The conditions of insolubility of the temperature field and uniqueness of the displacement field are also necessary while going round each inclusion:

$$\int_{L'_p}\left[\frac{\partial T}{\partial y}\right]_h\mathrm{d}\xi=0,\quad\int_{a_p^-}^{a_p^+}\left[u'_x\right]_h\mathrm{d}\xi=U\left(a_p^+\right)-U\left(a_p^-\right),\quad\int_{a_p^-}^{a_p^+}\left[u'_y\right]_h\mathrm{d}\xi=V\left(a_p^+\right)-V\left(a_p^-\right). \tag{3.18}$$

It is necessary to add the condition of equality to zero of the main moment of the forces applied to the inclusion for the determination of an additional rigid turn of the inclusion $\varepsilon_{\mathrm{B}}^{w}$ to the IC (3.15):

$$-\int_{a_p^-}^{a_p^+}\left\{x\left[\sigma_{yy\mathrm{B}}\right]_h+h\left\langle\sigma_{xy\mathrm{B}}\right\rangle_h\right\}\mathrm{d}\xi=M\left(a_p^+\right)-M\left(a_p^-\right). \tag{3.19}$$

It is possible to construct simpler variants, considering the inclusion of thermally and mechanically orthotropic, heat-insulated, diathermic, absolutely pliable (crack), perfectly rigid and other materials on the basis of the obtained interaction conditions (3.15).

The thermomechanical interaction conditions of a thin elastic heat-active inclusion layer with matrix (3.15) are given for the whole range of mechanical properties of inclusion materials. They are generalizations of IC [5,6,10,21,22] and equivalent to the most general equations for an "intermediate layer" [9,11] and a "thin elastic layer" [3, Chap. 1, Sect. 4] . The advantage of the method of IC creation presented here is that it gives the possibility of a mechanical interpretation of each term in the construction of the model.

4 Method of CPMF in a piecewise homogeneous solid with inclusions

The thermomechanical equilibrium of a piecewise-homogeneous medium consisting of two dissimilar half-planes S_k $(k = 1, 2)$ with the mechanical and physical parameters E_{*k}, ν_{*k}, α_{T*k}, λ_k is considered (Fig. 1). On the interface of the materials $L = L' \cup L''$ (coincides with the x-axis of a system of Cartesian co-ordinates) along $L' = \cup_{p=1}^{N} L'_p$, $L'_p = [a_p^-;\ a_p^+]$, N thin heat-active inclusions with parameters E_{B}, ν_{B}, α_{B}, λ_{B} are situated. The medium is loaded by heat fluxes, q_{xk}^{∞}, q_y^{∞}, and homogeneous tensions $\sigma_{yy}^{\infty} = p$, $\sigma_{xxk}^{\infty} = p_k$, $\sigma_{xy}^{\infty} = \tau$ acting at infinity; heat sources q_k, heat doublets q_{kk}, forces P_k and moments M_k situated at points z_{*k} of the matrix. A perfect or imperfect thermal contact between matrix and inclusions is assumed, perfect—between half-planes; heat $Q = \sum_{p=1}^{N} Q_p$ is generated from the region of the inclusion; $Q_p = 2h \int_{L'_p} Q_0^c(\xi)\mathrm{d}\xi$ is the flow of heat from the pth inclusion. The mechanical contact between the components of the system is assumed to be perfect.

Let us assume that the power factors do not disturb the temperature fields (coupled displacements and temperature processes are not taken into consideration). Thus, the problem of heat conductivity is first to examine the temperature-field disturbance by a thin inclusion and other temperature factors (by heat fluxes, sources and doublets). After this the temperature field can be considered as an external loading (by the homogeneous solution $(\sigma_{kj}^0,\ u_j^0\ (k, j \to x, y))$ in the thermoelasticity problem.

4.1 Problem of heat conductivity (plane temperature field)

Let us consider perfect thermal contact between half-planes

$$[T]_{L''} = 0, \quad [\lambda \partial T/\partial y]_{L''} = 0, \quad \left[t^0\right]_L = 0, \quad \left[\lambda \partial t^0/\partial y\right]_L = 0 \quad \big([f]_L \equiv f^-(x) - f^+(x), \quad x \in L\big). \tag{4.1}$$

Hence JFM with respect to heat-conduction problems suggests the simulation of the thin inclusion by a distribution of heat sources and doublets (jumps of temperature and heat flux) in its middle surface

$$\begin{aligned} &\left[\frac{\partial \hat{t}}{\partial x}\right]_L = f_5(x), \quad \left[\lambda \frac{\partial \hat{t}}{\partial y}\right] = f_6(x), \quad \big(x \in L'\big), \\ &f_r(x) = 0 \quad (r = 5, 6), \quad \big(x \in L''\big). \end{aligned} \tag{4.2}$$

When heat transfer from the lateral surfaces is absent, the temperature field is a harmonic function and it can be given as the real part of an analytic function, the complex potential of the temperature field:

$$T(z) = \mathfrak{Re}\Phi_T(z); \tag{4.3}$$

$$\frac{\partial T(z)}{\partial x} = \mathfrak{Re}\mathfrak{I}(z), \quad \frac{\partial T(z)}{\partial y} = -\mathfrak{Im}\mathfrak{I}(z), \quad \Phi'_T(z) = \Im(z). \tag{4.4}$$

According to (2.2) and (4.4), we have [10]:

$$\Im(z) = \Im^0(z) + \hat{\Im}(z), \quad \Phi_T(z) = \Phi_T^0(z) + \hat{\Phi}_T(z), \tag{4.5}$$

$$\begin{aligned} \Phi_T^0(z) = -\frac{1}{2\pi}\Bigg\{&\frac{2\pi \overline{q^{\infty}}}{\lambda_k} z + D_{k1}(z) + \big(p_k - p_j\big)\left\{\overline{D}_{k1}(z) + \overline{D}_{k2}(z)\right\} \\ &+ D_{k2}(z) + 2p_j\left\{D_{j1}(z) + D_{j2}(z)\right\} + \Phi_T^{\infty}\Bigg\}, \quad D_{k1}(z) = \frac{q_k}{\lambda_k}\log(z - z_{*k}), \\ D_{k2}(z) = -\frac{q_{kk}}{\lambda_k}&\frac{\exp(i\theta_k)}{z - z_{*k}}, \quad p_k = p\lambda_k, \quad p = \frac{1}{\lambda_1 + \lambda_2}, \quad c = p_k\lambda_j = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}, \\ &\Phi_T^{\infty} = \mathrm{const} \quad (z \in S_k;\ k = 1, 2;\ j = 3 - k). \end{aligned} \tag{4.6}$$

From (4.1) for $|x| \to \infty$ we have an additional relation between the heat fluxes acting at infinity: $\lambda_2 q_{x1}^{\infty} = \lambda_1 q_{x2}^{\infty}$. Using (4.3)–(4.6), from (4.2) it is easy to derive two Riemann–Hilbert problems [23, Chap. 6], the solution of which looks like

$$\hat{\Im}(z) = -2p_j H_5(z) + 2ipH_6(z) \quad (z \in S_k;\ k = 1, 2;\ j = 3 - k), \tag{4.7}$$

$$\hat{\Phi}_T(x) = 2p_j \tilde{H}_5(z) - 2ip\tilde{H}_6(z) \quad (z \in S_k;\ k = 1, 2;\ j = 3 - k). \tag{4.8}$$

Here

$$\begin{gathered} \left\{H_r(z),\ \tilde{H}_r(z)\right\} = \frac{1}{2\pi i}\int_{L'}\left\{\frac{1}{\xi - z},\ \log(\xi - z)\right\} f_r(\xi)\,\mathrm{d}\xi, \quad (z \in S_k), \\ \tilde{f}_r(x) = \int f_r(t)dt,\ \left\{H_r(x),\ \tilde{H}_r(x)\right\} = \frac{1}{\pi}\int_{L'}\left\{\frac{1}{\xi - x},\ \log|\xi - x|\right\} f_r(\xi)\,\mathrm{d}\xi, \quad \left(x \in L'\right). \end{gathered} \tag{4.9}$$

Thus, as a result of (4.3)–(4.9), the distribution of the temperature field in an arbitrary point of the matrix by means of the unknown JF f_r $(r = 5, 6)$ is found.

For vanishing and uniqueness at infinity of the temperature field, an additional condition is necessary

$$\int_{L'} f_6(\xi)\,\mathrm{d}\xi + \sum_{k=1}^{2} q_k = 0, \tag{4.10}$$

which follows from (4.6) and (4.8). Comparison of (4.10) with the condition of heat balance in the region of each inclusion (3.16) indicates that, for vanishing and uniqueness of the temperature field at infinity, an implementation of a heat-balance condition in the medium is necessary: $Q + \sum_{k=1}^{2} q_k = 0$.

We use (4.3), (4.8), using (4.1), (4.2) and the Sokhotski–Plemelj formulas [23, Chap. 6], to obtain the boundary values of the disturbed part of the temperature and its derivatives on the interface. Their substitution in IC (3.15) leads to a system of singular integral equations (SSIE) for the determination of JF f_r $(r = 5, 6)$:

$$\begin{cases} \rho_{51}H_5(x) + \rho_{52}s_{5n}(x) + \rho_{55}f_6(x) = F_5(x), \quad \left(x \in L_n';\ n = \overline{1, N}\right) \\ \rho_{61}H_6(x) + \rho_{62}H_5'(x) + \rho_{63}s_{6n}(x) + \rho_{64}f_5(x) + \rho_{65}f_6'(x) = F_6(x), \end{cases} \tag{4.11}$$

where

$$\begin{gathered} \rho_{51} = -2c\left(h + \lambda_{\mathrm{B}}\alpha_y^+\right), \quad \rho_{55} = \lambda_{\mathrm{B}}\alpha_y^- - (p_2 - p_1)\left(h + \lambda_{\mathrm{B}}\alpha_y^+\right), \\ \rho_{52} = \lambda_{\mathrm{B}}, \quad \rho_{61} = 2\lambda_{\mathrm{B}}ph, \quad \rho_{62} = -2c\lambda_{\mathrm{B}}\alpha_y^- h, \quad \rho_{63} = -1, \quad \rho_{64} = \lambda_{\mathrm{B}}(p_2 - p_1)h, \\ \rho_{65} = \lambda_{\mathrm{B}}h\left(\alpha_y^+ - (p_2 - p_1)\alpha_y^-\right), \quad s_{rn}(x) = \int_{a_n^-}^{x} f_r(\xi)\mathrm{d}\xi, \quad \alpha_y^{\pm} = \frac{\alpha_{y2} \pm \alpha_{y1}}{2\alpha_{y1}\alpha_{y2}}, \end{gathered}$$

where α_{y1}, α_{y2} are heat-transfer coefficients between matrix and inclusion or terms opposite to the coefficients of heat resistance; the functions $F_5(x)$, $F_6(x)$ are shown in Appendix 1.

The functions $f_r(x)$ satisfy the additional conditions

$$\int_{L_n'} f_r(\xi)\mathrm{d}\xi = Q_n^r\left(n = \overline{1, N};\ r = 5, 6\right), \quad Q_n^5 = 0,\ Q_n^6 = \hat{Q}_x\left(a_n^+\right) - \hat{Q}_x\left(a_n^-\right) + Q_n.$$

The first among these (for $r = 5$) determines the continuity of the temperature field; the second $(r = 6)$ follows from the heat balance.

The following partial cases of the SSIE (4.11) deserve special attention:

4.1.1 Perfect thermal contact of inclusion with matrix

Passing to the limit $\alpha_{yk} \to \infty$ $(k = 1, 2)$ in (4.11), we obtain equations that describe the case of perfect thermal contact between matrix and inclusions on the interface of the half-planes

$$\begin{cases} \rho_{51}H_5(x) + \rho_{52}s_{5n}(x) + \rho_{55}f_6(x) = F_5(x), \\ \rho_{61}H_6(x) + \rho_{63}s_{6n}(x) + \rho_{64}f_5(x) = F_6(x)\left(x \in L_n';\ n = \overline{1, N}\right), \end{cases} \tag{4.12}$$

where now

$$\rho_{51} = -2ch, \quad \rho_{52} = \lambda_{\mathrm{B}}, \quad \rho_{55} = -(p_2 - p_1)h, \quad \rho_{61} = 2\lambda_{\mathrm{B}}ph, \quad \rho_{63} = -1, \quad \rho_{64} = \lambda_{\mathrm{B}}(p_2 - p_1)h.$$

4.1.2 Homogeneous matrix

When $\lambda_1 = \lambda_2 = \lambda$, $\alpha_{y1} = \alpha_{y2} = \alpha_y$, we obtain a separated system of equations for an inclusion in a homogeneous matrix from (4.11):

$$-\lambda\left(h + \lambda_{\mathrm{B}}/\alpha_y\right) H_5(x) + \lambda_{\mathrm{B}} s_{5n}(x) = F_5(x),$$
$$(\lambda_{\mathrm{B}} h/\lambda) H_6(x) + \left(\lambda_{\mathrm{B}} h/\alpha_y\right) f_6'(x) - s_{6n} = F_6(x) \quad \left(x \in L_n';\ n = \overline{1, N}\right).$$

4.1.3 Heat-insulated inclusion (HII)

This case is obtained from (4.11) either by setting $\alpha_{y1} = \alpha_{y2} = 0$ or by setting $\lambda_{\mathrm{B}} = 0$. As a result, we have an expression for $f_6(x)$ and a singular integral equation:

$$f_6(x) = -\left[\lambda \frac{\partial t^0}{\partial y}\right]_h, \quad H_5(x) + \frac{\lambda_2 - \lambda_1}{2\lambda_1\lambda_2} f_6(x) = \frac{\lambda_2 + \lambda_1}{2\lambda_1\lambda_2}\left\langle \lambda \frac{\partial t^0}{\partial y}\right\rangle_h \quad \left(x \in L_n';\ n = \overline{1, N}\right)$$

which can be solved analytically [24, Chap. 4, Sect. 3, Chap. 6].

4.1.4 Diathermic inclusion (DI)

Considering $\alpha_{y1} = \alpha_{y2} = \alpha_y$ and passing to the limit $\lambda_{\mathrm{B}} \to \infty$ in (4.11), we will get

$$\tilde{H}_5(x) + \frac{1}{\alpha_y} f_6(x) = -\left[t^0\right]_h - \frac{1}{\alpha_y}\left[\lambda \frac{\partial t^0}{\partial y}\right]_h, \quad \left(x \in L_n';\ n = \overline{1, N}\right)$$
$$2pH_6(x) + (p_2 - p_1) f_5(x) + \frac{c}{\alpha_y}\frac{\partial}{\partial x}\left\{2H_5(x) - (p_2 - p_1) f_6(x)\right\} = -\frac{\partial}{\partial x}\left\{\left\langle t^0\right\rangle_h + \frac{1}{\alpha_y}\left\langle \lambda \frac{\partial t^0}{\partial y}\right\rangle_h\right\}.$$

These expressions may be presented as conditions of imperfect contact of half-planes along L':

$$[T]_h + \frac{1}{\alpha_y}\left[\lambda \frac{\partial T}{\partial y}\right]_h = 0, \quad \frac{\partial}{\partial x}\left\{\langle T\rangle_h + \frac{1}{\alpha_y}\left\langle \lambda \frac{\partial T}{\partial y}\right\rangle_h\right\} = 0.$$

A similar problem was studied in [25, Chap. 2].

4.1.5 Physically equivalent materials

Considering $\lambda_1 = \lambda_2 = \lambda_{\mathrm{B}} = \lambda$ in (4.11), we have $F_5(x) = F_6(x) = 0$. Consequently, the disturbed field is absent (the JF is equal to zero) in the uniqueness result of the solution.

For a more detailed analysis let us consider the case of perfect thermal contact of a single inclusion with the matrix. A study of the characteristic part of SSIE (4.12) proves that JF can be given as:

$$f_5^m(x) = \frac{q}{a\sqrt{\lambda_1\lambda_2}}(\varphi_1(x) + \varphi_2(x)), \quad f_6^m(x) = \frac{q}{a}(\varphi_1(x) - \varphi_2(x)),$$
$$\varphi_k(x) = \left(a_m^+ - x\right)^{-1/2\pm\eta}\left(x - a_m^-\right)^{-1/2\mp\eta}\varphi_k^*(x), \quad k = \left\{\begin{matrix}1\\2\end{matrix}\right\}, \tag{4.13}$$

where $2a = a_m^+ - a_m^-$, $\varphi_k^*(x)$ are dimensionless regular functions; q denotes the intensity of the thermal loading of the medium; the term η is determined by the characteristic part of the SSIE (4.12) [24, Chap. 6, Sect. 3] and equals $\eta = \arctan\left(0.5(\lambda_2 - \lambda_1)/\sqrt{\lambda_2\lambda_1}\right)$.

By introduction of a polar co-ordinate system (r, θ) with the origin near the right or the left tip of the inclusion $z = \pm r \cdot \exp(i\theta) \pm a_m^\pm$ (Fig. 1), it is possible to obtain two-term asymptotic expressions for the distribution of the temperature gradients in the vicinity of the tips ($|z_1| << 2a$) with (2.1), (4.7)–(4.9), (4.13) taken into account:

$$\left\{\begin{array}{c}\dfrac{\partial}{\partial x}\left(T-T^0\right)\\ \dfrac{\partial}{\partial y}\left(T-T^0\right)\end{array}\right\}=\frac{1}{2\sqrt{a\pi\lambda_1\lambda_2}}\left[\left(\frac{r}{2a}\right)^{-\frac{1}{2}\pm\eta}k_1^{\pm}M^{\mp}(-1)+\left(\frac{r}{2a}\right)^{-\frac{1}{2}\mp\eta}k_2^{\pm}M^{\pm}(1)\right]$$

$$+\frac{pq}{a}\left\{\begin{array}{c}N_6^{\pm}\\ -\dfrac{\lambda_j}{\sqrt{\lambda_1\lambda_2}}N_3^{\pm}\end{array}\right\}+O\left(r^{1/2\mp\eta}\right),\quad M^{\mp}(r)=\left\{\begin{array}{c}-\dfrac{\lambda_j}{\sqrt{\lambda_1\lambda_2}}\sin\theta^{\mp}+r\cos\theta^{\mp}\\ r\sin\theta^{\mp}+\dfrac{\lambda_j}{\sqrt{\lambda_1\lambda_2}}\cos\theta^{\mp}\end{array}\right\},$$

$$\theta^{\mp}=\frac{\theta}{2}\mp\eta\theta,\quad z\in S_k\ (k=1,2;\ j=3-k),$$

$$N_r^{\pm}=\lim_{x\to a_m^{\pm}\mp 0}\left[H_r^m(x)+\tan\pi\eta\ f_{11-r}^m(x)\right]=\frac{1}{2\cos\pi\eta}\sum_{n=1}^{\infty}\frac{(\pm1)^n\,\Gamma\left(n+\frac{1}{2}\mp\eta\right)}{(n-1)!\Gamma\left(\frac{3}{2}\mp\eta\right)}A_n^r\quad (r=5,6);\tag{4.14}$$

$$A_n^r=\left\{\begin{array}{l}A_n^1-A_n^2\ (r=6),\\ A_n^1+A_n^2\ (r=5),\end{array}\right.\quad \text{if}\quad \varphi_m^*(x)=\sum_{n=0}^{\infty}A_n^m P_n^{-1/2\pm\eta,-1/2\mp\eta}(x);$$

where $k_1^{\pm}, k_2^{\pm}$ denote gradients of the temperature-field intensity factors (GTFIF), which are defined by the correlations

$$\lim_{\substack{r\to0,\\ \theta=0}}2\sqrt{\pi a\lambda_1\lambda_2}\left\{\frac{\partial T}{\partial x}\left(a_m^{\pm}\right)-\mathrm{i}\frac{\partial T}{\partial y}\left(a_m^{\pm}\right)\right\}=\left(\frac{r}{2a}\right)^{-\frac{1}{2}\pm\eta}L_j(-1)k_1^{\pm}+\left(\frac{r}{2a}\right)^{-\frac{1}{2}\mp\eta}L_j(1)k_2^{\pm},$$

$$k_m^{\pm}=\frac{q\sqrt{\pi\lambda_1\lambda_2}}{(\lambda_1+\lambda_2)\sqrt{a}\cos\pi\eta}\varphi_m^*(\pm1),\quad L_j(r)=\left(r-i\frac{\lambda_j}{\sqrt{\lambda_1\lambda_2}}\right),\quad z\in S_k(k=1,2;\ j=3-k).$$

The coefficients $N_r^{\pm}$ describe the second terms and, as it appears, they are determined only by the homogeneous solution in the limiting cases of HII and DI.

In partial cases of physical equivalence of matrix materials, either DI or HII, the characteristic parameter $\eta=0$ and the asymptotic expressions can be simplified:

$$\lim_{\substack{r\to0,\\ \theta=0}}2\lambda\sqrt{\pi a}\left\{\frac{\partial T}{\partial x}\left(a_m^{\pm}\right)-i\frac{\partial T}{\partial y}\left(a_m^{\pm}\right)\right\}=\left(\frac{r}{2a}\right)^{-\frac{1}{2}}\left(k_x^{\pm}-ik_y^{\pm}\right),\quad \left(\begin{array}{l}k_x^{\pm}=k_2^{\pm}-k_1^{\pm},\\ k_y^{\pm}=k_2^{\pm}+k_1^{\pm}\end{array}\right).$$

$$\left\{\begin{array}{c}\dfrac{\partial}{\partial x}\left(T-T^0\right)\\ \dfrac{\partial}{\partial y}\left(T-T^0\right)\end{array}\right\}=\frac{k_x^{\pm}}{\lambda\sqrt{2\pi r}}\left\{\begin{array}{c}\cos\dfrac{\theta}{2}\\ \sin\dfrac{\theta}{2}\end{array}\right\}+\frac{k_y^{\pm}}{\lambda\sqrt{2\pi r}}\left\{\begin{array}{c}-\sin\dfrac{\theta}{2}\\ \cos\dfrac{\theta}{2}\end{array}\right\}-\frac{q}{4a\lambda}\left\{\begin{array}{c}n_6^{\pm}\\ -n_5^{\pm}\end{array}\right\}+O\left(\sqrt{r}\right),$$

$$\lambda\left(T-T_0\right)=\pm\sqrt{2r/\pi}\left(k_y^{\pm}\sin\frac{\theta}{2}+k_x^{\pm}\cos\frac{\theta}{2}\right)+\left[n_6^{\pm}\left(a+r\cos\theta\right)-n_5^{\pm}r\sin\theta\right]+O\left(r^{3/2}\right).$$

In particular, for a loading with heat sources of strength $q_2=-q_1=q$ in the points $z_2=\bar{z}_1=iy^*$:

$$k_x^{\pm}=0,\quad k_y^{\pm}=-\frac{q}{\sqrt{\pi a}}\left[1+\left(y_*/a\right)^2\right]^{-1/2}\quad \text{for HII,}\tag{4.15}$$

$$k_x^{\pm}=0,\quad k_y^{\pm}=0\quad \text{for DI.}\tag{4.16}$$

It is remarkable that GTFIF under such loading do not depend on the physical characteristics.

Using (4.15), (4.16) we easily obtain GTFIF for a semi-infinite interface defect by introducing a local system of co-ordinates $\xi=z-a$ starting near the right tip of the inclusion and passing to the limit $a\to\infty$.

4.2 Thermoelasticity problem in the case of a plane temperature field

Vectors of stresses and displacements from the homogeneous state are continuous on the interface L:

$$\left(\sigma^0_{yy1} - \mathrm{i}\sigma^0_{xy1}\right) - \left(\sigma^0_{yy2} - \mathrm{i}\sigma^0_{xy2}\right) = 0,\ \left({u^0_{x1}}' + \mathrm{i}{u^0_{y1}}'\right) - \left({u^0_{x2}}' + \mathrm{i}{u^0_{y2}}'\right) = 0\ \ (x \in L). \tag{4.17}$$

For a disturbed state four JF are introduced similar to (4.2):

$$\begin{aligned}&\left(\hat{\sigma}_{yy1} - \mathrm{i}\hat{\sigma}_{xy1}\right) - \left(\hat{\sigma}_{yy2} - \mathrm{i}\hat{\sigma}_{xy2}\right) = f_1(x) - \mathrm{i}\, f_2(x) = f_1^*(x),\\ &\left(\hat{u}'_{x1} + \mathrm{i}\hat{u}'_{y1}\right) - \left(\hat{u}'_{x2} + \mathrm{i}\hat{u}'_{y2}\right) = f_3(x) + \mathrm{i}\, f_4(x) = f_3^*(x)\quad (x \in L),\end{aligned} \tag{4.18}$$

whence $f_r(x) = 0$ $(r = 1, 2, 3, 4)$ when $x \in L''$.

The stresses and displacements in each of the considered half-planes S_k can be written in terms of complex Kolosov–Muskhelishvili potentials $\Phi_k(z),\ \Psi_k(z)$. According to (2.1) let us represent CPMF as the sum of homogeneous, $\Phi^0_k(z)$, and disturbed, $\hat{\Phi}_k(z)$, solutions, assuming that the temperature field is now an external loading. Using analytic continuation [23, Chap. 6], it is possible to express the homogeneous solution as follows:

$$\begin{aligned}&\sigma^0_{yyk}(z) - \mathrm{i}\sigma^0_{xyk}(z) = \Phi^0_k(z) - \Phi^0_k(\bar{z}) + (z - \bar{z})\,\overline{\Phi^{0\prime}_k(z)},\\ &2G_k\left[{u^0_{xk}}'(z) + i{u^0_{yk}}'(z)\right] = \kappa_k\Phi^0_k(z) + \Phi^0_k(\bar{z}) - (z - \bar{z})\,\overline{\Phi^{0\prime}_k(z)}\\ &\quad + \beta_{Tk}\Phi_T(z),\quad \kappa_k = \frac{3 - \nu_{k^*}}{1 + \nu_{k^*}},\quad \beta_{Tk} = 2\alpha_{T*k}G_k\quad (z \in S_k;\ k = 1, 2),\end{aligned} \tag{4.19}$$

where the potential $\Phi_T(z)$ is determined by (4.5), (4.6), (4.8).

The potential of the homogeneous field, $\Phi^0_k(z)$, may be represented in the form [19; 23, Chap. 6]:

$$\begin{aligned}&\Phi^0_k(z) = \Gamma_k + S_{11}(z) + S_{22}(z) + \beta_{Tk}\phi_k\left[D_{k1}(z) + D_{k2}(z)\right] + \varepsilon_{kj}D_{j1}(z) + \Phi^0_{0k}(z),\\ &\Phi^0_j(z) = -\bar{\Gamma}_j - \bar{\Gamma}'_j - \overline{S_{jj}}(z) - \overline{R_{jj}}(z) - \overline{R_j}(z) - z\overline{S'_{11}}(z) - \beta_{Tj}\phi_j\sum_{m=1}^{4}\bar{D}_{jm}(z)\\ &\qquad -\varepsilon_{jk}\left[\bar{D}_{k1}(z) + z\bar{D}'_{k1}(z)\right] - \beta_{Tj}\phi_j\left[z\bar{D}'_{j1}(z) + z\bar{D}'_{j2}(z)\right] + \Phi^0_{0j}(z),\ (z \in S_k;\ \ j = 3 - k);\\ &D_{k3}(z) = -\frac{q_k}{\lambda_k}\frac{\bar{z}_{*k}}{z - z_{*k}},\quad D_{k4}(z) = -\frac{q_{kk}}{\lambda_k}\left[\frac{\exp(-\mathrm{i}\theta_k)}{z - z_{*k}} + \frac{\bar{z}_{*k}\exp(\mathrm{i}\theta_k)}{(z - z_{*k})^2}\right],\\ &\quad\Gamma_k = \frac{p + p_k}{4},\quad \Gamma'_k = \frac{p - p_k}{2} + \mathrm{i}\tau,\quad S_{kj}(z) = -\frac{\phi_k P_j}{z - z_{*j}},\quad R_j(z) = \frac{-\mathrm{i}M_j}{2\pi\left(z - z_{*j}\right)^2},\\ &R_{kj}(z) = \phi_k\left[\frac{\kappa_k\overline{P_j}}{z - z_{*j}} - \frac{\bar{z}P_j}{\left(z - z_{*j}\right)^2}\right],\quad \phi_k = \frac{1}{2\pi\left(1 + \kappa_k\right)},\quad \varepsilon_{kj} = -\beta_{Tk}\phi_k\frac{q_k\lambda_j}{q_j\lambda_k}.\end{aligned} \tag{4.20}$$

Here the functions $\Phi^0_{0m}(z)$ vanish at infinity and are holomorphic in S_k $\ (k, m = 1, 2)$. The constant ε_{kj} is determined from the condition of uniqueness of stresses and displacements at infinity. Implementation of condition (4.10) of the general heat balance in the medium is thus required.

Application of the method used in [23, Chap. 6] gives

$$\begin{aligned}
\Phi_{0k}^{0}(z) &= m_k^k S_{jj}(z) + \left(m_k^j - 1\right)\left[\overline{S_{kk}}(z) + \overline{R_{kk}}(z) + \overline{R_k}(z) + z\overline{S_{kk}'}(z)\right] \\
&\quad + \left(m_k^j - 1\right)\beta_{Tk}\phi_k\left[\bar{D}_{k3}(z) + z\bar{D}_{k1}'(z) - \frac{q_k}{\lambda_k}\right] + \left(2\beta_{Tk}\phi_k m_k^j p_j - \varepsilon_{kj}\right)D_{j1}(z) \\
&\quad + \beta_{Tk}\phi_k\left(2m_k^j p_k - 1\right)\bar{D}_{k1}(z) - 2\pi\beta_{Tk}\phi_k m_k^j\hat{\Phi}_T(z), \\
\Phi_{0j}^{0}(z) &= \left(1 - m_k^k\right)S_{jj}(z) - m_k^j\left[\overline{S_{kk}}(z) + \overline{R_{kk}}(z) + \overline{R_k}(z) + z\overline{S_{kk}'}(z)\right] \\
&\quad - m_k^j\beta_{Tk}\phi_k\bar{D}_{k3}(z) - \left(\beta_{Tj}\phi_j m_k^j - \varepsilon_{jk}\right)\left[z\bar{D}_{k1}'(z) - \frac{q_k}{\lambda_k}\right] \\
&\quad - \left(2\beta_{Tk}\phi_k m_k^j p_j - \beta_{Tj}\phi_j\right)D_{j1}(z) - \left(2\beta_{Tk}\phi_k m_k^j p_k - \varepsilon_{jk}\right)\bar{D}_{k1}(z) \\
&\quad + 2\pi\beta_{Tk}\phi_k m_k^j\hat{\Phi}_T(z), \quad m_k^j = \frac{4E_{*j}}{e_{kj}}, \quad p_k = \frac{\lambda_k}{\lambda_1 + \lambda_2}, \\
e_{kj} &= E_{*k}\left(1 + \nu_{*j}\right) + E_{*j}\left(3 - \nu_{*k}\right), \quad (z \in S_k; \quad k = 1, 2; \quad j = 3 - k)
\end{aligned} \tag{4.21}$$

which, along with (4.19), (4.20), enable to specify the homogeneous solution $\sigma_{kj}^0,\ u_j^0$.

It can be proved that the following loading conditions at infinity are needed to solve the problem:

$$\left(\kappa_1\Gamma_1 - \bar{\Gamma}_1 - \bar{\Gamma}_1' + \beta_{T1}\Phi_T^\infty\right)G_2 = \left(\kappa_2\Gamma_2 - \bar{\Gamma}_2 - \bar{\Gamma}_2' + \beta_{T2}\Phi_T^\infty\right)G_1,$$

$$\Gamma_k = \frac{1}{4}(p + p_k), \quad \Gamma_k' = \frac{1}{2}(p - p_k) + \mathrm{i}\tau,$$

$\alpha_{T2}/\lambda_2 = \alpha_{T1}/\lambda_1$; when $q_{xk}^\infty = 0 \quad (k = 1, 2)$ and $\alpha_{T2} = \alpha_{T1}$ when $q_y^\infty = 0$.

Substitution of (4.17)–(4.21) in the IC (3.15) (without taking into account terms of order h^2) leads to SSIE:

$$\begin{cases} \rho_{k1}H_2(x) + \rho_{k2}H_4(x) + \rho_{k3}s_{2n}(x) + \rho_{k4}s_{4n}(x) + \rho_{k5}f_1(x) + \rho_{k6}f_3(x) = F_k(x), \\ \rho_{j1}H_1(x) + \rho_{j2}H_3(x) + \rho_{j3}s_{1n}(x) + \rho_{j4}s_{3n}(x) + \rho_{j5}f_2(x) + \rho_{j6}f_4(x) = F_j(x), \end{cases} \tag{4.22}$$

$$\left(x \in L_n';\ n = \overline{1, N}; \quad k = 1, 2; \quad j = 3, 4\right),$$

where the functions $F_m(x)$ $(m = 1, 2, 3, 4)$ are given in Appendix A and the coefficients are equal to:

$$\begin{aligned}
&\rho_{11} = 2m_{12}^{-}h/E_{*\mathrm{B}}, \quad \rho_{12} = -2l_1^{+}h/E_{*\mathrm{B}}, \quad \rho_{13} = -\nu_{*\mathrm{B}}/E_{*\mathrm{B}}, \quad \rho_{14} = 1, \\
&\rho_{15} = \left(m_{12}^{+} - m_{21}^{+}\right)h/E_{*\mathrm{B}}, \quad \rho_{16} = 2l_1^{-}h/E_{*\mathrm{B}}, \quad \rho_{21} = -2h\left(\nu_{*\mathrm{B}}m_{12}^{-}/E_{*\mathrm{B}} + l_2^{+}\right), \\
&\rho_{22} = 2h\left(\nu_{*\mathrm{B}}l_1^{+}/E_{*\mathrm{B}} - m_{12}^{-}\right), \quad \rho_{24} = 0, \quad \rho_{25} = -h\left(\left(m_{12}^{+} - m_{21}^{+}\right)\nu_{*\mathrm{B}}/E_{*\mathrm{B}} + 2l_2^{-}\right), \\
&\rho_{26} = \left(m_{12}^{+} - m_{21}^{+} - 2\nu_{*\mathrm{B}}l_1^{-}/E_{*\mathrm{B}}\right)h, \quad \rho_{23} = -1/E_{*\mathrm{B}}, \quad \rho_{31} = 2\nu_{*\mathrm{B}}m_{12}^{-}h/E_{*\mathrm{B}}, \\
&\rho_{32} = 2\nu_{*\mathrm{B}}l_1^{+}h/E_{*\mathrm{B}}, \quad \rho_{33} = 1/\left(\nu_{*\mathrm{B}}E_{*\mathrm{B}}\right), \quad \rho_{35} = -\nu_{*\mathrm{B}}\left(m_{12}^{+} - m_{21}^{+}\right)h/E_{*\mathrm{B}}, \\
&\rho_{36} = 2\nu_{*\mathrm{B}}l_1^{-}h/E_{*\mathrm{B}}, \quad \rho_{34} = -1, \quad \rho_{41} = -2m_{12}^{-}h, \quad \rho_{42} = 2l_2^{+}h, \quad \rho_{44} = -1, \\
&\rho_{43} = -2(1 + \nu_{*\mathrm{B}})/E_{*\mathrm{B}}, \quad \rho_{45} = -2l_2^{-}h, \quad \rho_{46} = -\left(m_{12}^{+} - m_{21}^{+}\right)h, \\
&m_{kj}^{\pm} = \frac{E_{*k}}{2}\left(\frac{1 + \nu_{*j}}{e_{kj}} \pm \frac{3 - \nu_{*j}}{e_{jk}}\right), \quad l_1^{\pm} = \frac{E_{*1}E_{*2}}{2}\left(\frac{1}{e_{12}} \pm \frac{1}{e_{21}}\right), \\
&l_2^{\pm} = \frac{(3 - \nu_{*1})(1 + \nu_{*2})}{e_{12}} \pm \frac{(3 - \nu_{*2})(1 + \nu_{*1})}{e_{21}} \ (k = 1, 2; \quad j = 3 - k)
\end{aligned}$$

Additional conditions, (3.16)–(3.19), must be thus satisfied.

An investigation [10] of the characteristic part of SSIE (4.22) proves that JF f_n^*, $n = \{1, 3\}$ can be given as $f_n^*(x) = (a_m^+ - x)^{-1/2 \pm \mathrm{i}\mu_n}(x - a_m^-)^{-1/2 \mp \mathrm{i}\mu_n}\varphi_n^*(x)$, where $\varphi_n^*(x) = \sum_{k=0}^\infty A_k^n P_k^{-1/2 \pm \mathrm{i}\mu_n, -1/2 \mp \mathrm{i}\mu_n}(x)$ are dimensionless regular functions, and $\mu_1 = \frac{1}{2\pi}\log\frac{\kappa_1 e_{21}}{\kappa_2 e_{12}}$, $\mu_3 = \frac{1}{2\pi}\log\frac{e_{21}}{e_{12}}$.

Introduction of a system of polar co-ordinates (r, θ) with the origin near the right or the left tip of the inclusion $z = \pm r \cdot \exp(\mathrm{i}\theta) \pm a_m^\pm$ (Fig. 1), makes it possible to obtain two-term asymptotic expressions for the distribution of the stresses and displacements in the vicinity of the tips $\left(|z_1| << 2a, 2a = a_m^+ - a_m^-\right)$ [26] with (4.19)–(4.21) taken into account:

$$\|\sigma_{yyk},\ \sigma_{xxk},\ \sigma_{xyk},\ \sigma_{rrk},\ \sigma_{\theta\theta k},\ \sigma_{r\theta k}\|^{\mathbf{T}} = \frac{1}{2\sqrt{2\pi r}} \sum_{m=1}^{2} \Big(L_{1km}^{\pm}(\theta, \mu_m) \cdot K_{1,m}^{\pm} + L_{2km}^{\pm}(\theta, \mu_m) \cdot K_{2,m}^{\pm} \Big) + N_k \left(\pm a_m^{\pm}\right) + O\left(r^{2/3}\right), \quad z \in S_k, \tag{4.23}$$

where $L_{1km}^{\pm}, L_{2km}^{\pm};\ N_k\left(\pm a_m^{\pm}\right)$ are coefficients defined in [26]; $K_{1,m}^{\pm},\ \ K_{2,m}^{\pm}\ (m = 1,\ 2)$ denote the generalized stress-intensity factors (GSIF), which are defined by the correlations

$$K_{1,m}^{\pm} - \mathrm{i}K_{2,m}^{\pm} = \lim_{r\to 0\ (\theta=0)} \sqrt{2\pi r} \cdot \exp\left(\mathrm{i}\varepsilon n\right)\left(\sigma_{yy} - \mathrm{i}\sigma_{xy}\right).$$

$$\varepsilon_n = \mu_n \log\frac{r}{2a} \quad (m = 1,\ 2;\ n = -2m + 5).$$

The following special cases of SSIE (4.22) are important for analysis:

4.2.1 Perfectly rigid inclusion

For $E_{\mathrm{B}} \to \infty,\ \ \alpha_{Tk} = 0$:

$$f_3^*(x) = -\left[u_x^{0\prime}(x) + \mathrm{i}u_y^{0\prime}(x)\right]_h \approx 0, \quad H_1^*(x) - \mathrm{i}\beta_1 f_1^*(x) = F_1^*(x), \tag{4.24}$$

$$F_1^*(x) \equiv \frac{\mathrm{i}}{2l_2^+}\left\langle u_x^{0\prime}(x) + \mathrm{i}u_y^{0\prime}(x)\right\rangle_h - \frac{\varepsilon_{\mathrm{B}}}{l_2^+} - \mathrm{i}\frac{m_{21}^+ - m_{12}^+}{l_2^+}\left[u_x^{0\prime}(x) + \mathrm{i}u_y^{0\prime}(x)\right]_h + \frac{\mathrm{i}m_{21}^-}{\pi l_2^+}\int_{L'}\left[u_x^{0\prime}(\xi) + \mathrm{i}u_y^{0\prime}(\xi)\right]_h \frac{\mathrm{d}\xi}{\xi - x}, \quad \beta_1 = \frac{l_2^-}{l_2^+} \quad \left(x \in L'\right).$$

4.2.2 Cracks

When $E_{\mathrm{B}} \to \infty$, one may derive from (4.22):

$$f_1^*(x) = -\left[\sigma_{yy}^0(x) - \mathrm{i}\sigma_{xy}^0(x)\right]_h \approx 0, \quad H_3^*(x) - \mathrm{i}\beta_3 f_3^*(x) = F_3^*(x), \tag{4.25}$$

$$F_3^*(x) \equiv \frac{\mathrm{i}}{2l_1^+}\left\langle \sigma_{yy}^{0\prime}(x) - \mathrm{i}\sigma_{xy}^{0\prime}(x)\right\rangle_h + \mathrm{i}\frac{m_{21}^+ - m_{12}^+}{l_1^+}\left[\sigma_{yy}^0(x) - \mathrm{i}\sigma_{xy}^0(x)\right]_h + \frac{m_{21}^-}{\pi l_1^+}\int_{L'}\left[\sigma_{yy}^0(\xi) - \mathrm{i}\sigma_{xy}^0(\xi)\right]_h \frac{\mathrm{d}\xi}{\xi - x}, \quad \beta_3 = \frac{l_1^-}{l_1^+} \quad \left(x \in L'\right).$$

4.2.3 Piecewise-homogeneous matrix without inclusion

If $h \to 0$, then $f_r(x) = 0 \quad (r = 1, 2, 3, 4)$ and the disturbed solution is zero.

4.2.4 Homogeneous matrix

If $E_1 = E_2 = E,\ \ \nu_1 = \nu_2 = \nu,\ \ \alpha_{T1} = \alpha_{T2} = \alpha_T$, then $\rho_{m5} = \rho_{m6} = 0$ and SSIE (4.22) is simplified, separating into two independent systems of two equations each:

$$\begin{aligned} &\rho_{k1}H_2(x) + \rho_{k2}H_4(x) + \rho_{k3}s_{2n}(x)\,\rho_{k4}s_{4n}(x) = F_k(x),\\ &\rho_{j1}H_1(x) + \rho_{j2}H_3(x) + \rho_{j3}s_{1n}(x)\,\rho_{j4}s_{3n}(x) = F_j(x), \quad \left(x \in L_n';\ n = \overline{1, N};\ k = 1, 2;\ j = 3, 4\right). \end{aligned} \tag{4.26}$$

Then $\mu_m = 0\,(m = 1, 3)$: the solutions for a crack or perfectly rigid strip belong to the class of functions $\prod^{-1/2,-1/2}\left(f_m^*\right)$ and that is why they are mechanically correct.

4.2.5 Homogeneous medium without inclusion

This case can be obtained from (4.26), either by considering $h \to 0$ or $E \to E_{\mathrm{B}}, \nu \to \nu_{\mathrm{B}},\ \alpha_T \to \alpha_{T\mathrm{B}}, \lambda \to \lambda_{\mathrm{B}}$.

4.3 Thermoelasticity problem in the case of a generalized plane temperature field

From a geometrical point of view the problem for a piecewise-homogeneous plate in the case of a generalized plane temperature field is identical to the problem for a medium with a plane temperature field, when its lateral surfaces are heat-insulated. The plate is under the same external loading, except for a heat flux at infinity, for which the heat-transfer result has no physical meaning. The solution of this problem is reduced to the solution of the previous problem for a plane temperature field.

The influence of heat transfer from the lateral surfaces of a plate demands taking into account the thickness 2δ of the plate and the inclusions, their heat-transfer coefficients $\alpha_k, \alpha_{\mathrm{B}n}$, the ambient temperature T_C and the assumption about the symmetry of the action of all temperature factors to a median surface of the plate.

The stresses and displacements in each of the parts S_k $(k = 1, 2)$ must satisfy all constitutive relations (3.2), (3.3) and the equation of displacement compatibility. The Airy function $\Im_k(x, y)$ [23, Chap. 2, Sect. 2] satisfies the compatibility equation (3.4):

$$\Delta\Delta\Im_k(x, y) + \alpha_{Tk} E_k \Delta T(x, y) = 0 \quad (k = 1, 2). \tag{4.27}$$

In the previous case for a plane temperature field, the function $T(x, y)$ was harmonic and therefore $\Im_k(x, y)$ was biharmonic. This enabled the application of complex-function theory. In this case T satisfies the Helmholz equation of heat conduction (3.6). That is why we write $\Im_k(x, y)$ as the sum of its biharmonic constituent $\Im_k^b(x, y)$ and the partial solution of the heterogeneous equation (4.27) reads: $\Im_k(x, y) = \Im_k^b(x, y) + \Im_k^T(x, y)$.

Assuming the function $\Im_k^T(x, y)$ to be known, let us define

$$\begin{aligned}
&\sigma_{xxk}^b(x, y) = \sigma_{xxk}(x, y) - \frac{\partial^2 \Im_k^T(x, y)}{\partial y^2}, \quad \sigma_{yyk}^b(x, y) = \sigma_{yyk}(x, y) - \frac{\partial^2 \Im_k^T(x, y)}{\partial x^2}, \\
&\sigma_{xyk}^b(x, y) = \sigma_{xyk}(x, y) + \frac{\partial^2 \Im_k^T(x, y)}{\partial x \partial y} \quad ((x, y) \sim z \in S_k;\ k = 1, 2).
\end{aligned} \tag{4.28}$$

Thus, taking into account (3.3), we have

$$\begin{aligned}
&\frac{\partial u_{xk}^b(x, y)}{\partial x} = \frac{\partial u_{xk}(x, y)}{\partial x} - \frac{\alpha_{Tk}\lambda_k\delta}{\alpha_k}\left(\frac{\partial^2 \Im_k^T(x, y)}{\partial y^2} - \nu_k \frac{\partial^2 \Im_k^T(x, y)}{\partial x^2}\right) - \alpha_{Tk} T, \\
&\frac{\partial u_{yk}^b(x, y)}{\partial y} = \frac{\partial u_{yk}(x, y)}{\partial y} - \frac{\alpha_{Tk}\lambda_k\delta}{\alpha_k}\left(\frac{\partial^2 \Im_k^T(x, y)}{\partial x^2} - \nu_k \frac{\partial^2 \Im_k^T(x, y)}{\partial y^2}\right) - \alpha_{Tk} T, \\
&\frac{\partial u_{xk}^b(x, y)}{\partial y} + \frac{\partial u_{yk}^b(x, y)}{\partial x} = \frac{\partial u_{xk}(x, y)}{\partial y} + \frac{\partial u_{yk}(x, y)}{\partial x} + \frac{2(1 + \nu_k)\alpha_{Tk}\lambda_k\delta}{\alpha_k}\frac{\partial^2 \Im_k^T(x, y)}{\partial x \partial y}.
\end{aligned} \tag{4.29}$$

Note that the expressions for $\sigma_{xxk}^b,\ \sigma_{yyk}^b,\ \sigma_{xyk}^b,\ u_{xk}^b,\ u_{yk}^b$ satisfy (3.2), (3.3) and the compatibility equation regarding the displacements in the regions S_k $(k = 1, 2)$. That is why, from a mathematical point of view, the determination of the "biharmonic" stresses and displacements, as introduced by (4.28), (4.29), is equivalent to the thermoelasticity problem for the case of a plane temperature field.

One may derive the partial solution $\Im_k^T(x, y)$ by integrating (4.27) using (3.6) [27]:

$$\Im_k^T(x, y) = -\frac{\alpha_{Tk} E_k \lambda_k \delta}{\alpha_k}\left[T(x, y) - T_c\right] + \frac{\alpha_{Tk} E_k}{2\alpha_k} W_k(x, y), \tag{4.30}$$

where, in the case of sources and doublets of heat action in the point z_{*k}, we have

$$W_k(x, y) = -\frac{q_k}{2\pi}\log r_{*k} + \frac{q_{kk}}{2\pi}\frac{(x - x_{*k})\cos\theta_k + (y - y_{*k})\sin\theta_k}{r_{*k}^2}, \quad r_{*k}^2 = \sqrt{(x - x_{*k})^2 + (y - y_{*k})^2}.$$

Thus, having the partial solution (4.30), by substitution of (4.28), (4.29) the "biharmonic" stresses and displacements are defined. Determination of these values is fully similar to the solution of a thermoelastic problem for a plane temperature field.

To illustrate this method, let us analyze the case of a homogeneous plate with an inclusion**.** Considering $E_1 = E_2 = E,\ \nu_1 = \nu_2 = \nu,\ \alpha_{T1} = \alpha_{T2} = \alpha_T,\ \lambda_1 = \lambda_2 = \lambda,\ \alpha_1 = \alpha_2 = \alpha$ and taking into account that, from a heat-conduction point of view, the inclusion can be represented by a line of distributed heat sources and doublets, one obtains an expression for $\Im_k^T(x, y)$, using (4.28):

$$\begin{aligned}\Im_k^T(x, y) = &-\frac{\alpha_T E}{\omega^2}\left\{\frac{1}{4\pi\lambda\delta}\sum_{k=1}^{2} q_k\left[K_0(\omega r_{*k}) + \log r_{*k}\right] + \frac{1}{4\pi\lambda\delta}\sum_{k=1}^{2} q_{kk}\left[(x - x_{*k})\cos\theta_k\right.\right.\\ &\left.+ (y - y_{*k})\sin\theta_k\right]\left[\frac{\omega K_1(\omega r_{*k})}{r_{*k}} - \frac{1}{r_{*k}^2}\right] - \frac{y}{2\pi}\int_{L'}\tilde{f}_5(\xi)\left[\frac{\omega K_1(\omega r)}{r} - \frac{1}{r^2}\right]\mathrm{d}\xi\\ &+\frac{1}{2\pi\lambda}\int_{L'}\tilde{f}_6(\xi)\left[K_0(\omega r) + \log r\right]\mathrm{d}\xi, \quad r = \sqrt{(x-\xi)^2 + y^2}.\end{aligned}$$

This function fully takes into account the temperature influence on the field of stresses and displacements.

Using this we obtain

$$\begin{aligned}&\sigma_{yyk}^{b0}(z) - \mathrm{i}\sigma_{xyk}^{b0}(z) = \Phi_k^0(z) - \Phi_k^0(\bar{z}) + (z - \bar{z})\overline{\Phi_k^{0\prime}(z)},\\ &2G\left[\hat{u}_{xk}^{b0\prime}(z) + \mathrm{i}\hat{u}_{yk}^{b0\prime}(z)\right] = \kappa\Phi_k^0(z) + \Phi_k^0(\bar{z}) - (z - \bar{z})\overline{\Phi_k^{0\prime}(z)},\\ &\hat{\sigma}_{yyk}^{b}(z) - \mathrm{i}\hat{\sigma}_{xyk}^{b}(z) = \hat{\Phi}_k(z) - \hat{\Phi}_k(\bar{z}) + (z - \bar{z})\overline{\hat{\Phi}_k'(z)},\\ &2G\left[\hat{u}_{xk}^{b\prime}(z) + \mathrm{i}\hat{u}_{yk}^{b\prime}(z)\right] = \kappa\hat{\Phi}_k(z) + \hat{\Phi}_k(\bar{z}) - (z - \bar{z})\overline{\hat{\Phi}_k'(z)} \quad (z \in S_k),\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}&\Phi_k^0(z) = \Gamma_k + S_{11}(z) + S_{22}(z),\\ &\Phi_j^0(z) = -\bar{\Gamma}_j - \bar{\Gamma}_j' - \overline{S_{jj}}(z) - \overline{R_{jj}}(z) - \overline{R_j}(z) - z\overline{S_{11}'}(z),\\ &\hat{\Phi}_k(z) = \frac{\mathrm{i}}{2(1+\kappa)}\left[H_1^*(z) + 2GH_3^*(z)\right],\\ &\hat{\Phi}_j(z) = \frac{\mathrm{i}}{2(1+\kappa)}\left[\kappa H_1^*(z) + 2GH_3^*(z)\right].\end{aligned} \tag{4.32}$$

Then by using Sokhotski–Plemelj formulas [23, Chap. 6], we obtain

$$\begin{aligned}&\left\langle\hat{\sigma}_{yy}^b - \mathrm{i}\hat{\sigma}_{xy}^b\right\rangle_L = -4\mathrm{i}\frac{1-\kappa}{1+\kappa}H_1^*(x) - 16\mathrm{i}\frac{G}{1+\kappa}H_3^*(x),\\ &\left[\hat{\sigma}_{yy}^b - \mathrm{i}\hat{\sigma}_{xy}^b\right]_L = f_1^*(x), \quad \left[\frac{\partial\hat{u}_x^b}{\partial x} + \mathrm{i}\frac{\partial\hat{u}_y^b}{\partial x}\right]_L = f_3^*(x),\\ &\left\langle\frac{\partial\hat{u}_x^b}{\partial x} + \frac{\partial\hat{u}_y^b}{\partial x}\right\rangle_L = -4\mathrm{i}\frac{\kappa}{G(1+\kappa)}H_1^*(x) + 4\mathrm{i}\frac{1-\kappa}{1+\kappa}H_3^*(x).\end{aligned} \tag{4.33}$$

Passing in (4.31)–(4.33) to real stresses and displacements, using (4.28), (4.29), we obtain the SSIE:

$$\begin{aligned}&\rho_{k1}H_2(x) + \rho_{k2}H_4(x) + \rho_{k3}s_{2n}(x) + \rho_{k4}s_{4n}(x) = F_k^T(x),\\ &\rho_{j1}H_1(x) + \rho_{j2}H_3(x) + \rho_{j3}s_{1n}(x) + \rho_{j4}s_{3n}(x) = F_j^T(x), \quad \left(x \in L_n';\ \ n = \overline{1, N};\ \ k = 1, 2;\ \ j = 3, 4\right),\end{aligned}$$

which differs from (4.26) only by the parts on the right. The additional terms (3.18), (3.19), having a similar physical meaning, must then be calculated.

Thus, the solution of the thermoelastic problem for a piecewise-homogeneous plate with heat transfer from the lateral surfaces is easily found from the known solution for a piecewise-homogeneous medium having a plane temperature field.

The methodology based on the inclusion model (3.10), (3.11), interaction conditions (3.15)–(3.19) and governed SSIE (4.11), (4.22) could be used to solve such problems as: the interaction between a number of interface inclusions; the interaction of macro- and micro-defects [4,28]; periodical arrays of inclusions [8]; the effect of combined thermal and mechanical loads [29,30] etc. As the PCCDD gives the external asymptotic solution, the only limitation of the proposed methodology is that the solution is exact in the entire solid, except in the vicinity of the heterogeneity (singularity). To analyze the CPMF at the tips of the defects, asymptotic methods are provided [31].

A detailed analysis of these problems will be done by the authors in later publications. Now we concentrate only on investigations of the thermoelastic state of a medium or a plate with heat-insulated lateral surfaces and a thin elastic heat-active interface inclusion.

5 Examples

The method described above has been used to study a piecewise-homogeneous matrix with a thin defect placed along $L' = [-a, a]$ for two cases of temperature loading: (1) by heat sources of intensity $q_k = \pm q$ in the points $z_{*k} = \pm \mathrm{i}d$; $k = (1, 2)$; (2) by heat doublets of intensity q_{kk} and identical orientation $\theta_k = \pi/2$ in these points. Using (4.24), (4.25), we may obtain closed-form solutions for an interface defect.

An absolutely pliable heat-insulated inclusion (crack) for the case $\alpha_{T*2}/\lambda_2 = \alpha_{T*1}/\lambda_1$ gives the following GSIF:

$$K^{\pm}_{1,1} - \mathrm{i}K^{\pm}_{2,1} = \pm\,\mathrm{i}\frac{qE_{*1}E_{*2}\alpha_{T*2}\sqrt{a/\pi}}{\lambda_2}\left\{\frac{1}{e_{21}}J^{\pm}_{22}(d/a,\ \mu_3) + \frac{1}{e_{12}}J^{\pm}_{21}(d/a,\ \mu_3)\right\}$$
$$\mp\frac{l^{+}_{1}}{\sqrt{\pi a}}\left(A^3_1 + \mathrm{i}A^4_1\right),\quad \mu_3 = \frac{1}{2\pi}\log\frac{e_{21}}{e_{12}},\quad \text{(loading 1)}; \tag{5.1}$$

$$K^{\pm}_{1,1} - \mathrm{i}K^{\pm}_{2,1} = \mp i\,\frac{(q_{22}E_{*2} + q_{11}E_{*1})\,\alpha_{T*2}}{16\sqrt{\pi a}\lambda_2}J^{\pm}_{61}(d/a,\ \mu_3)$$
$$-\frac{(\mathrm{i}\pm 2\mu_3)\,\alpha_{T*2}}{8\sqrt{\pi a}\lambda_2}\left\{q_{22}E_{*2}J^{\pm}_{62}(d/a,\ \mu_3) - q_{11}E_{*1}J^{\pm}_{63}(d/a,\ \mu_3)\right\}$$
$$\mp\mathrm{i}\frac{2l^{+}_{1}\alpha_{T*2}\,(\lambda_1 q_{22} + \lambda_2 q_{11})}{\sqrt{\pi a}\lambda_2\,(\lambda_1+\lambda_2)}J_{64}(d/a) \mp \mathrm{i}\frac{l^{+}_{1}}{\sqrt{\pi a}}\left(A^3_1 + iA^4_1\right),\quad \text{(loading 2)}. \tag{5.2}$$

If the half-planes the material properties are identical, and (5.1) becomes the simpler:

$$K^{\pm}_{1,1} - \mathrm{i}K^{\pm}_{2,1} = \mp\mathrm{i}\frac{qE_{*}\alpha_{T*}a\sqrt{a}}{2\lambda\sqrt{\pi}\sqrt{d^2+a^2}} \mp \frac{E_*}{4\sqrt{\pi a}}\left(A^3_1 + \mathrm{i}A^4_1\right)\quad \text{(loading 1)} \tag{5.3}$$

which differs from a similar expression in [32, Chap. 9, Sect. 1] only by the last term.

An absolutely pliable diathermic inclusion for the case $\alpha_{T*2} = \alpha_{T*1}$ yields:

$$K^{\pm}_{1,1} - \mathrm{i}K^{\pm}_{2,1} = \pm\,\mathrm{i}qE_{*1}E_{*2}\alpha_{T*}\sqrt{a/\pi}\left\{\frac{1}{\lambda_1 e_{12}}J^{\pm}_{23}(d/a, \mu_3) + \frac{1}{\lambda_2 e_{21}}J^{\pm}_{24}(d/a, \mu_3)\right\}$$
$$\mp\frac{\mathrm{i}l^{+}_{1}}{\sqrt{\pi a}}\left(A^3_1 + \mathrm{i}A^4_1\right),\quad \mu_3 = \frac{1}{2\pi}\log\frac{e_{21}}{e_{12}},\quad \text{(loading 1)}; \tag{5.4}$$

$$K^{\pm}_{1,1} - \mathrm{i}K^{\pm}_{2,1} = \mp\mathrm{i}\frac{\alpha_{T*2}}{16\sqrt{\pi a}}\left(\frac{q_{22}E_{*2}}{q_2} + \frac{q_{11}E_{*1}}{q_1}\right)J^{\pm}_{61}(d/a, \mu_3) - \frac{(\mathrm{i}\pm 2\mu_4)\,\alpha_{T*2}}{8\sqrt{\pi a}}$$
$$\times\left\{\frac{q_{22}E_{2*}}{\lambda_2}J^{\pm}_{62}(d/a, \mu_3) - \frac{q_{11}E_{1*}}{\lambda_1}J^{\pm}_{63}(d/a, \mu_3)\right\} \mp \mathrm{i}\frac{l^{+}_{1}}{\sqrt{\pi a}}\left(A^3_1 + \mathrm{i}A^4_1\right),\quad \text{(loading 2)}. \tag{5.5}$$

For a perfectly rigid heat-insulated strip in the case $\alpha_{T*2}G_2/(\lambda_2\kappa_2) = \alpha_{T*1}G_1/(\lambda_1\kappa_1)$ we have:

$$K_{1,2}^{\pm} - \mathrm{i}K_{2,2}^{\pm} = \mp\mathrm{i}\frac{2q\sqrt{a/\pi}m_{21}^{-}\alpha_{T*2}E_{*2}}{l_2^{+}\lambda_2(3-\nu_{*2})}\left\{\frac{(3-\nu_{*1})(1+\nu_{*2})}{e_{12}}J_{21}^{\pm}(d/a,\mu_1) + \frac{(3-\nu_{*2})(1+\nu_{*1})}{e_{21}}\right.$$
$$\left.\times J_{22}^{\pm}(d/a,\mu_1)\right\} \mp \mathrm{i}\frac{m_{21}^{-}}{\sqrt{\pi a}}\left(A_1^1 - \mathrm{i}A_1^2\right), \quad \mu_1 = \frac{1}{2\pi}\log\frac{\kappa_1 e_{21}}{\kappa_2 e_{12}}, \quad \text{(loading 1);} \tag{5.6}$$

$$K_{1,2}^{\pm} - \mathrm{i}K_{2,2}^{\pm} = \pm\mathrm{i}\frac{m_{21}^{-}}{2\sqrt{\pi a}l_2^{+}}\left\{d_1 J_{65}^{\pm}(d/a,\mu_1) + d_2 J_{66}^{\pm}(d/a,\mu_1)\right\}$$
$$+ \frac{(\mathrm{i}\pm 2\mu_1)m_{kj}^{-}}{8\sqrt{\pi a}l_2^{+}}\left\{\frac{\alpha_{T*1}(1+\nu_{1*})q_{11}}{\lambda_1}J_{63}^{\pm}(d/a,\mu_1) - \frac{\alpha_{T*2}(1+\nu_{2*})q_{22}}{\lambda_2}J_{62}^{\pm}(d/a,\mu_1)\right\}$$
$$\pm\mathrm{i}\frac{(1+\nu_{1*})(1+\nu_{2*})m_{21}^{-}l_3^{+}}{\sqrt{\pi a}l_2^{+}}\left(\frac{q_{11}}{\lambda_1} + \frac{q_{22}}{\lambda_2}\right)J_{64}(d/a) \mp \mathrm{i}\frac{m_{21}^{-}}{\sqrt{\pi a}}\left(A_1^1 - \mathrm{i}A_1^2\right), \quad \text{(loading 2).} \tag{5.7}$$

A perfectly rigid diathermic strip in the case $\alpha_{T*2}G_2/\kappa_2 = \alpha_{T*1}G_1/\kappa_1$ gives:

$$K_{1,2}^{\pm} - \mathrm{i}K_{2,2}^{\pm} = \mp\mathrm{i}\frac{2q\sqrt{a/\pi}m_{21}^{-}\alpha_{T*2}E_{*2}}{l_2^{+}(3-\nu_{*2})}\left\{\frac{(3-\nu_{*1})(1+\nu_{*2})}{\lambda_1 e_{12}}J_{23}^{\pm}(d/a,\mu_1)\right.$$
$$\left.+ \frac{(3-\nu_{*2})(1+\nu_{*1})}{\lambda_1 e_{21}}J_{24}^{\pm}(d/a,\mu_1)\right\} \mp \mathrm{i}\frac{m_{21}^{-}}{\sqrt{\pi a}}\left(A_1^1 - \mathrm{i}A_1^2\right), \quad \text{(loading 1);} \tag{5.8}$$

$$K_{1,2}^{\pm} - \mathrm{i}K_{2,2}^{\pm} = \pm\mathrm{i}\frac{m_{21}^{-}}{2\sqrt{\pi a}l_2^{+}}\left\{d_1 J_{65}^{\pm}(d/a,\mu_1) + d_2 J_{66}^{\pm}(d/a,\mu_1) + \frac{(\mathrm{i}\pm 2\mu_1)m_{kj}^{-}}{8\sqrt{\pi a}l_2^{+}}\right.$$
$$\left.\times\left\{\frac{\alpha_{T*1}(1+\nu_{*1})q_{11}}{\lambda_1}J_{63}^{\pm}(d/a,\mu_1) - \frac{\alpha_{T*2}(1+\nu_{*2})q_{22}}{\lambda_2}J_{62}^{\pm}(d/a,\mu_1)\right\}\mathrm{i}\frac{m_{21}^{-}}{\sqrt{\pi a}}\left(A_1^1 - iA_1^2\right),\right.$$
$$\text{(loading 2).} \tag{5.9}$$

The notations used in (5.1)–(5.9) are given in Appendix B.

The dimensionless GSIF $K_{m,j}^{0}$ $(m, j = 1, 2)$ are given in Figs. 2–5 for some values of the dimensionless parameters $K = E_{*2}/E_{*1}$, y_*/a when $\nu_{*1} = \nu_{*2} = 0, 3$ for loading 1 (by heat sources).

The calculations show that GSIF $K_{2,m}$ $(m = 1, 2)$, generated by a loading in the direction normal to the axis of the inclusion line, are of maximum value when the materials of the half-planes are mechanically identical; GSIF $K_{1,m}$ $(m = 1, 2)$ caused by a shear on an inclusion surface achieve extreme values when the mechanical properties of the matrix materials differ approximately by a factor of ten.

It also turned out, that the use of doublets as the loading factors (at least in the limiting cases studied here) does not change a high-quality figure substantially in comparison with loading by heat sources. The difference is that the doublet influences the CPMF of the matrix closer to the inclusion than for a heat source.

5.1 Numerical analysis for a homogeneous matrix

Changes in GSIF in the case of an elastic inclusion of thickness $h/a = 0.001$ in a homogeneous matrix have been investigated numerically by the Lobatto–Chebyshev collocation technique [33] by use of Richardson extrapolation. The calculations were carried out when $\nu = \nu_{\mathrm{B}}$, $\alpha_{T\mathrm{B}}/\alpha_T = 0$ for two ways of temperature loading: (1) by heat sources of intensity $q_k = \pm q$ in the points $z_{*k} = \pm\mathrm{i}y_*$, $k = (1, 2)$; (2) by heat sources of the same intensity in the points $z_{*k} = \pm x_*$, $k = (1, 2)$ for a wide variety of the relative stiffness $k = E_{\mathrm{B}}/E$ and relative heat conductivity of the inclusion, $\Lambda = \lambda_{\mathrm{B}}/\lambda$ $(10^{-5} \le k,\ \Lambda \le 10^{5})$. To achieve a precision of order 1%, it was sufficient to retain 40

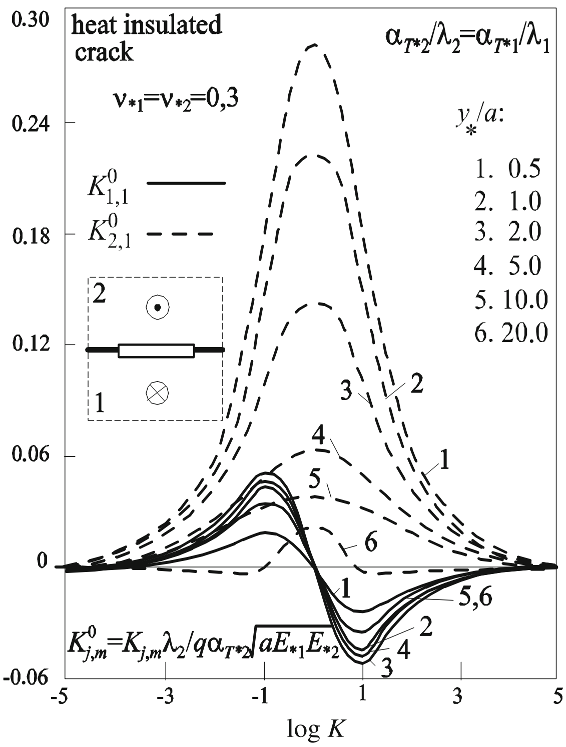

Fig. 2 The dimensionless generalized stress-intensity factors for a heat insulated crack

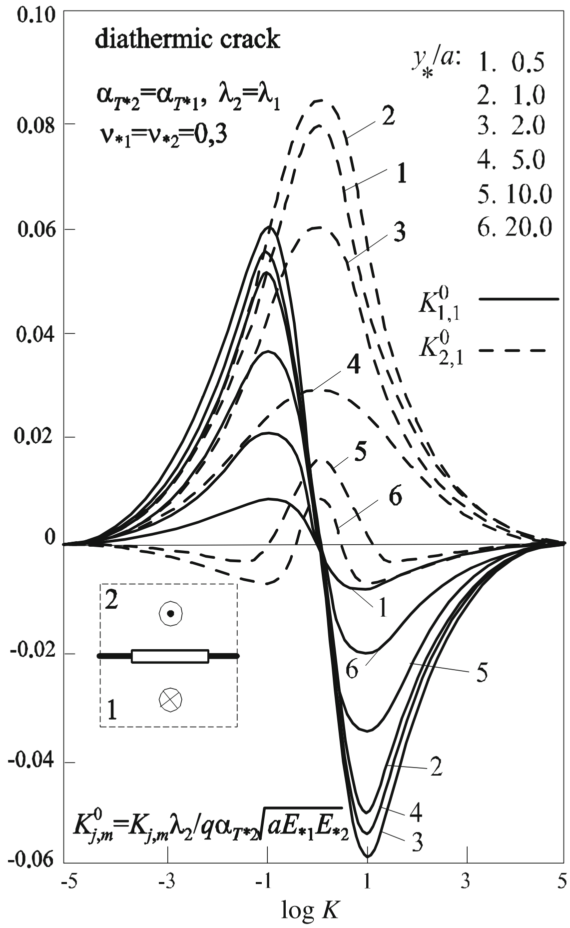

Fig. 3 The dimensionless generalized stress-intensity factors for a diathermic crack

terms in the decompositions of the unknown JF concerning the Chebyshev series with a distinguished square-root singularity.

Estimated values of the dimensionless GSIF $K^0_{j,m} = K_{j,m}\lambda / \left(q\alpha_{T*}E_*\sqrt{a}\right)$ $(j, m = 1, 2)$, depending on the dimensionless parameters k, Λ, y_*/a, x_*/a, are shown in Fig. 6.

Extreme values of GSIF are noticeable for the cases of a heat-insulated crack and an absolutely rigid diathermic inclusion. An increase of the relative heat-conduction coefficient Λ leads to a decrease of GSIF following an absolute value. The last effect generates also the relative removal of sources from the inclusion (Fig. 6). However, it should be noted that certain distant sources of the inclusion exist, when the absolute value of GSIF reaches its maximum.

It is also clear that the GSIF values, as obtained for $k = 10^{-5}$, $\Lambda = 10^{-5}$, for $k = 10^{-5}$, $\Lambda = 10^5$, for $k = 10^5$, $\Lambda = 10^{-5}$ and for $k = 10^5$, $\Lambda = 10^5$, differ from those obtained for an absolutely pliable heat-insulated, absolutely pliable diathermic, absolutely rigid heat-insulated and absolutely rigid diathermic inclusion, differing no more than 1%, which is within the limits of numerical precision. To achieve this precision, one must have $h/a \leq 0.001$. It is clear that the thinner the inclusion is, the more precise the result will be, as obtained on the basis of the jump-functions concept.

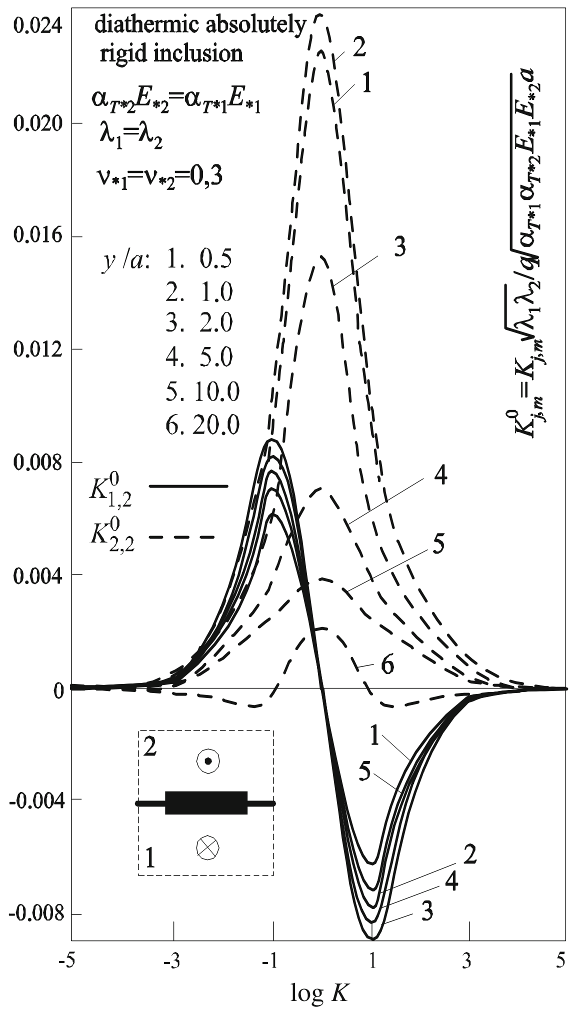

Fig. 4 The dimensionless generalized stress-intensity factors for a diathermic absolutely rigid inclusion

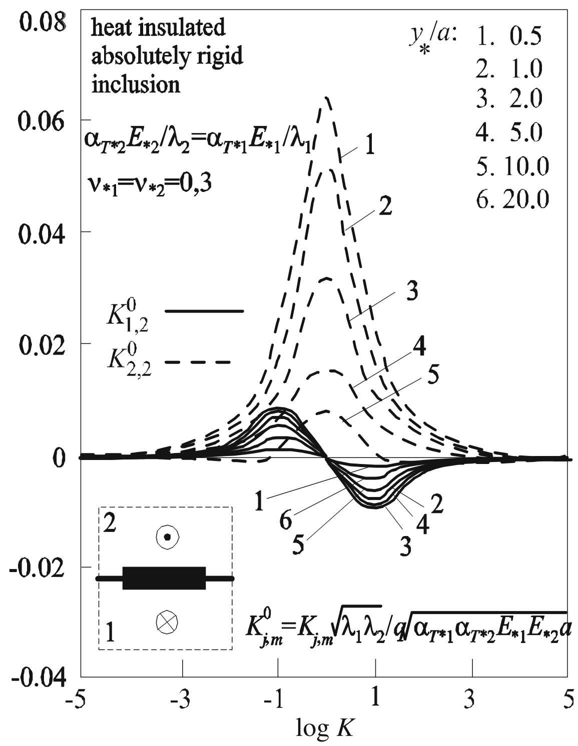

Fig. 5 The dimensionless generalized stress-intensity factors for a heat-insulated absolutely rigid inclusion

6 Conclusions

An effective method to model the presence of thin inclusions of arbitrary physical nature in bodies has been given. Using this method, we have reduced the plane thermoelastic problem for two bounded dissimilar half-planes with thin heat-active interface inclusions to two separate systems of singular integral equations, namely for a heat-conduction problem and for a thermoelasticity problem with a known temperature field. All possible limiting cases regarding the physical and elastic parameters have been obtained. For general properties of the inclusion material the system has been solved by a collocation method. The concept of generalized stress-intensity factors has been introduced and their dependence on the material properties and several ways of thermal loading have been analyzed. Most significant is the fact that the extreme values of generalized stress-intensity factors are obtained when the physical and elastic parameters of the inclusion are extreme. The generalized stress-intensity factors, generated by loadings normal to the axial line of the inclusion, are maximal in size when the matrix materials are identical mechanically, and generalized stress-intensity factors, caused by a shear on an inclusion surface lead to extreme values when the mechanical properties of the matrix materials differ approximately by a factor of ten.

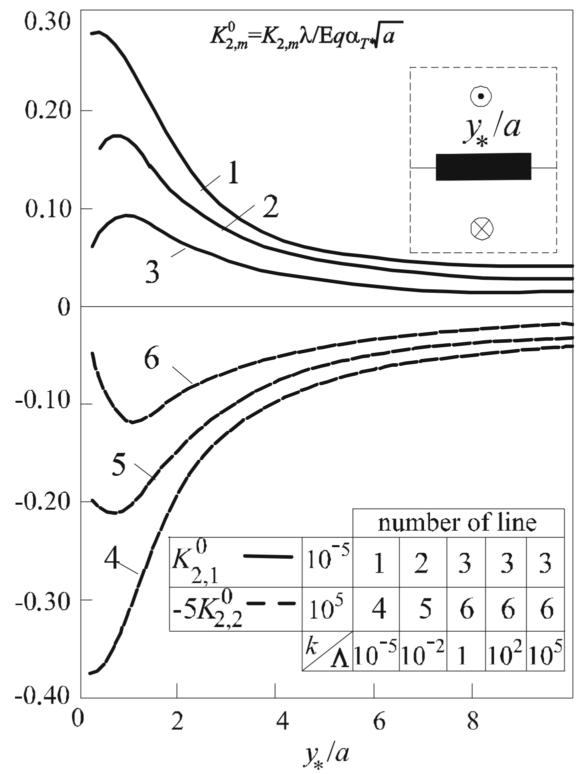

Fig. 6 Dimensionless generalized stress-intensity factors for a wide range of k and Λ in a homogeneous matrix

The method to reduce the solution of the thermoelastic problem for a piecewise-homogeneous plate with heat transfer from the lateral surfaces to the known solution for a piecewise-homogeneous medium having a plane temperature was discussed.

The only limitation of the proposed methodology is that the solution is valid everywhere, except in the vicinity of the tips of the heterogeneity (singularity), because the PCCDD gives the external asymptotic solution. To analyze the CPMF in the tips of the defects, asymptotic methods are provided.

The methodology based on the inclusion model (3.10), (3.11), interaction conditions (3.15)–(3.19) and governing SSIE (4.11), (4.22) can be useful for solving a number of similar problems. Examples are: the interaction between some of the interface inclusions; the interaction between macro- and microdefects; periodic arrays of inclusions; the effect of combined thermal and mechanical loads etc.

The authors believes that the discussed technique for modeling a thin inclusion together with a jump-function method can be an effective tool for analyzing plane piezoelectric problems for solids with thin inclusions.

Appendix A: The right parts of the thermomechanical interaction conditions of a thin elastic heat-active inclusion layer with matrix (3.15)

$$F_1(x) = \frac{\nu_B}{E_B} N_x\left(a_p^-\right) - V\left(a_p^-\right) + \frac{\nu_B}{E_B}\int_{a_p^-}^{x}\left[\sigma^0_{xy}\right]_h d\xi - \int_{a_p^-}^{x}\left[u_y^{0\prime}\right]_h d\xi - \frac{h}{E_B}\left\langle\sigma^0_{yy}\right\rangle_h$$
$$+\frac{h^2}{E_B}\frac{\partial}{\partial x}\left[\sigma^0_{xy}\right]_h - \alpha_{T}Bh\left\{\langle T\rangle_h + \left[\frac{\lambda}{\alpha_y}\frac{\partial T}{\partial y}\right]_h\right\} - \frac{\alpha_{TB}}{\lambda_B}h^2\left[\lambda\frac{\partial T}{\partial y}\right]_h,$$

$$F_2(x) = \frac{1}{E_B} N_x\left(a_p^-\right) + \frac{1}{E_B}\int_{a_p^-}^{x}\left[\sigma_{xy}^0\right]_h \mathrm{d}\xi - h\left\langle u_x^{0\prime}\right\rangle_h - \frac{\nu_B}{E_B} h\left\langle \sigma_{yy}^0\right\rangle_h + h^2\frac{\partial}{\partial x}\left[u_y^{0}{}'\right]_h$$
$$-\frac{2-\nu_B}{E_B} h^2\frac{\partial}{\partial x}\left[\sigma_{xy}^0\right]_h + \alpha_{TB} h\left\{\langle T\rangle_h + \left[\frac{\lambda}{\alpha_y}\frac{\partial T}{\partial y}\right]_h\right\} + \frac{\alpha_{TB}}{\lambda_B} h^2\left[\lambda\frac{\partial T}{\partial y}\right]_h,$$

$$F_3(x) = -\frac{\nu_B}{E_B} N_{xy}\left(a_p^-\right) + \int_{a_p^-}^{x}\left\{\left[u_x^{0\prime}\right]_h - \frac{1}{\nu_B E_B}\left[\sigma_{yy}^0\right]_h\right\}\mathrm{d}\xi - \frac{h^2}{E_B}\frac{\partial}{\partial x}\left[\sigma_{yy}^0\right]_h$$
$$+\frac{\nu_B}{E_B} h\left\langle\sigma_{xy}^0\right\rangle_h - \frac{1+\nu_B}{\nu_B}\alpha_{TB}\int_{a_p^-}^{x}\left\{[T]_h + \left\langle\frac{\lambda}{\alpha_y}\frac{\partial T}{\partial y}\right\rangle_h\right\}\mathrm{d}\xi - \alpha_{TB} h^2\frac{\partial}{\partial x}\left\{[T]_h + \left\langle\frac{\lambda}{\alpha_y}\frac{\partial T}{\partial y}\right\rangle_h\right\},$$

$$F_4(x) = V'\left(a_p^-\right) + \int_{a_p^-}^{x}\left[u_x^{0\prime}\right]_h\mathrm{d}\xi + \frac{1}{G_B}\int_{a_p^-}^{x}\left[\sigma_{yy}^0\right]_h\mathrm{d}\xi - h\left\langle u_y^{0\prime}\right\rangle_h - 2h\varepsilon_B$$

$$F_5(x) = \frac{\lambda_B}{h}\left(1-\frac{h^2}{2}\frac{\partial^2}{\partial x^2}\right)\left\{\left[t^0\right]_h + \left\langle\frac{\lambda}{\alpha_y}\frac{\partial t^0}{\partial y}\right\rangle_h\right\} - \left\langle\lambda\frac{\partial t^0}{\partial y}\right\rangle_h - \frac{h}{2\delta}\left[Q_B\right]_h,$$
$$F_6(x) = Q_x\left(a_p^-\right) - \lambda_B h\frac{\partial}{\partial x}\left\{\left\langle t^0\right\rangle_h + \left[\frac{\lambda}{\alpha_y}\frac{\partial t^0}{\partial y}\right]_h\right\} - h^2\frac{\partial}{\partial x}\left[\lambda\frac{\partial t^0}{\partial y}\right]_h + \int_{a_p^-}^{x}\left[\lambda\frac{\partial t^0}{\partial y}\right]_h\mathrm{d}\xi$$
$$-2h\int_{a_p^-}^{x} Q_B^C(\xi)\,\mathrm{d}\xi\quad\left(x\in L_p'\right).$$

Appendix B: The notations used in (5.1)–(5.9)

$$d_1 = \left[\frac{\alpha_{T*1}(1+\nu_{*1})}{4} + \alpha_{T*1}(p_2-p_1)\right]\frac{q_{11}}{2\lambda_1} + \left[\frac{\alpha_{T*2}(1+\nu_{*2})}{4} + 2\alpha_{T*1}p_2\right]\frac{q_{22}}{2\lambda_2},$$
$$d_2 = \left[\frac{\alpha_{T*2}(1+\nu_{*2})}{4} + \alpha_{T*2}(p_1-p_2)\right]\frac{q_{22}}{2\lambda_2} + \left[\frac{\alpha_{T*1}(1+\nu_{*1})}{4} + 2\alpha_{T*2}p_1\right]\frac{q_{11}}{2\lambda_1},$$

$$J_{2m}^{\pm}(y,\eta) = \frac{y^2 \pm \mathrm{i}\gamma_m y}{\sqrt{1+y^2}}\exp\left(2\gamma_m\eta\,\mathrm{arccot}\,y\right) - \sqrt{1+y^2}\quad(m=1,2),$$
$$J_{2m}^{\pm}(y,\eta) = \frac{y^2 \pm \mathrm{i}\gamma_m y}{\sqrt{1+y^2}}\exp\left(2\gamma_m\eta\,\mathrm{arccot}\,y\right) - y\quad(m=3,4),$$
$$J_{61}^{\pm}(y,\eta) = 1 - \frac{1}{\sqrt{1+y^2}}\cosh 2\eta\,\mathrm{arccot}\,y \mp \frac{\mathrm{i}}{\sqrt{1+y^2}}\mathrm{sh} 2\eta\,\mathrm{arccot}\,y,$$
$$J_{64}(y) = \frac{y}{\sqrt{1+y^2}} - 1,\quad J_{6m}^{\pm}(y,\eta) = \frac{\mathrm{i}y^2 \mp \gamma_m y}{\left(1+y^2\right)^{3/2}}\exp\left(2\gamma_m\eta\,\mathrm{arccot}\,y\right)\quad(m=2,3),$$
$$J_{6m}^{\pm}(y,\eta) = 1 - \frac{y\pm\mathrm{i}\gamma_m}{\sqrt{1+y^2}}\exp\left(2\gamma_m\eta\,\mathrm{arccot}\,y\right)\quad(m=5,6),\quad \gamma_m = (-1)^m.$$

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